Temporal and spatial aspects of correlation networks and dynamical network models: analytical approaches and physical applications

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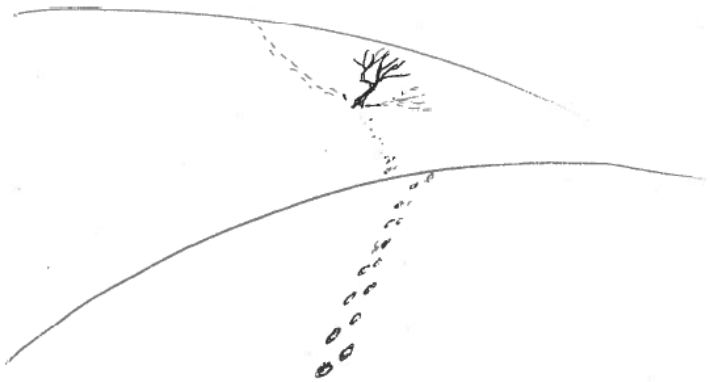

To my dear family, without whose love I would not be.

Abstract

Complex networks have been successfully applied to study phenomena in engineering, sociology and natural systems. In the thesis I studied the complex architectures of networks, the network evolution in time, the interpretation of the networks measures and a particular class of processes taking place on complex networks.

Firstly, I derived the measures to characterize temporal networks evolution in order to detect spatial variability patterns in evolving systems. Using introduced measures, I analyzed networks ensembles, constructed from temperature field variability for the Asian Monsoon and two-dimensional flow-driven dynamical system.

Secondly, I introduced a novel flow-network method to construct networks from flows, that also allows to modify the set-up from purely relying on the velocity field. This theoretical approach verifies relations between an underlying dynamics and a corresponding correlation matrix and correlation network measures, generalizes previous studies and overcomes the restriction to stationary flows. The flow-network method is developed for correlations of a scalar quantity (temperature, for example), which satisfies advection-diffusion dynamics in the presence of forcing and dissipation. The flow-network method for the time-series analysis analytically constructs correlation matrices and complex networks measures. This allows to characterize transport in the fluids, to identify various mixing regimes in the flow and to apply this method to advection-diffusion dynamics, data from climate and other systems, where particles transport plays a crucial role.

Thirdly, I developed a novel Heterogeneous Opinion-Status model (HOpS) and analytical technique to study dynamical processes on networks. Such processes may represent a diffusion of opinions or spread of diseases in society. The novel HOpS model of heterogeneous spreading on a network serves as an accessible test case for the methods for analysis of processes on networks, despite or rather exactly because of its relative simplicity. A new analytical techniques are based on properties of random walks and network topology. Surprisingly, a discrete phase space of the HOpS model has particular properties, which depend on characteristics of network topology and heterogeneity, i.e. distribution of nodes' statuses.

All in all, methods, derived in the thesis, allow to quantify evolution of various classes of complex systems, to get insight into physical meaning of correlation networks and analytically to analyze processes, taking place on networks.

Zusammenfassung

Komplexe Netzwerke wurden erfolgreich angewendet, um Phänomene in den Ingenieurwissenschaften, der Soziologie und natürlichen Systemen zu analysieren. In der vorliegenden Arbeit untersuchte ich die komplexen Strukturen von Netzwerken, deren zeitliche Entwicklung, die Interpretationen von verschieden Netzwerk-Massen und die Klassen der Prozesse darauf.
Als Erstes leitete ich Masse für die Charakterisierung der zeitlichen Entwicklung der Netzwerke her, um räumlich Veränderungsmuster zu erkennen. Die eingeführten Masse wendete ich auf Netzwerke der Temperaturfeld-Variabilität für den asiatischen Monsoon und ein zweidimensionales strömungsgetriebenes dynamisches System an.
Als Nächstes führe ich eine neue Methode zur Konstruktion komplexer Netzwerke von Flussfeldern ein, bei welcher man das Set-up auch rein unter Berufung Berufung auf das Geschwindigkeitsfeld ändern kann. Dies ist gleichzeitig ein Ansatz, um Beziehungen zwischen der Korrelationsmatrix und dem Klimanetzwerk zu überprüfen, welche verschiedene Klimanetzwerkmasse generalisiert und die Einschränkung auf stationäre Flüsse von früheren Arbeiten überkommt. Diese Verfahren wurden für die Korrelationen skalarer Grössen, z. B. Temperatur, entwickelt, welche eine Advektions-Diffusions-Dynamik in der Gegenwart von Zwingen und Dissipation. Die Flussnetzwerk-Methode zur Zeitreihenanalyse konstruiert die Korrelationsmatrizen und komplexen Netzwerke. Dies ermöglicht die Charakterisierung von Transport in Flüssigkeiten, die Identifikation verschiedene Misch-Regimes in dem Fluss und die Anwendung auf die Advektions-Diffusions-Dynamik, Klimadaten und anderen Systemen, in denen Teilchentransport eine entscheidende Rolle spielen.
Als Letztes, entwickelte ich ein neuartiges Heterogener Opinion Status Modell (HOpS) und Analysetechnik basiert auf Random Walks und Netzwerktopologie Theorien, um dynamischen Prozesse in Netzwerken zu studieren, wie die Verbreitung von Meinungen in sozialen Netzwerken oder Krankheiten in der Gesellschaft. Ein neues Modell heterogener Verbreitung auf einem Netzwerk wird als Beispielssystem für HOpS verwendent, um die vergleichsweise Einfachheit zu nutzen. Die Analyse eines diskreten Phasenraums des HOPS-Modells hat überraschende Eigenschaften, welches sensibel auf die Netzwerktopologie reagieren.

Die in dieser Arbeit entwickelten Methoden zum Aufbau von Netzwerken aus einem Fluss-Systems helfen dabei, einen Einblick in die physikalische Bedeutung der Klima Netze und deren Masse zu erhalten. Sie können verallgemeinert werden, um verschiedene Klassen von komplexen Netzwerken zu quantifizieren, Transportphänomene zu charakterisieren und verschiedene Zeitreihen zu analysieren.

Contents

List of Figures xi

List of Tables xvii

Introduction. 19

1 Research questions for the network theory and applications 19
- 1.1 Evolving networks: spatial and temporal properties 20
- 1.2 Theoretical analysis of correlation networks from the advection-diffusion dynamics 24
- 1.3 Dynamics on networks 25

Evolving networks: methods of analysis for random models and data structures 27

2 Introduction: static and evolving networks 27
- 2.1 General definitions from network theory 28
- 2.2 Network measures and characteristics 28
- 2.3 Basic types of networks 33
- 2.4 Functional networks 36

3 Methods to characterize evolving networks 42
- 3.1 Quantitative indices to estimate networks evolution 42
- 3.2 Qualitative measures to estimate networks evolution 44

4 Applications of temporal network measures to evolving networks . . 46
- 4.1 Index for random evolving networks 46
- 4.2 Network measures for evolving networks embedded in space . 47
- 4.3 Estimation of global variability for evolving networks 48

5 Discussions and conclusions 52

Theoretical foundation of correlation networks 57

6 Introduction to correlation networks 57
- 6.1 Correlation networks from data and dynamical systems . . . 57
- 6.2 Historical overview over network theory aspects for analysis of dynamical systems 58
- 6.3 Research questions: construction and analysis of correlation networks 60

7 Analysis of systems using flow-networks 62
- 7.1 Introduction to the flow-networks method 63
- 7.2 Algorithm of flow-networks construction 64

7.3 Calculation of correlations for advection-diffusion dynamics . 65
7.4 Correlation function for flow-networks 66
8 Results of flow-network analysis 67
8.1 Analytical results 67
8.2 Numerical results 70
8.3 A model flow setup for flow-networks 70
8.4 Centrality measures for flow-networks 72
9 Discussions 78
10 Conclusions and Future prospectives 80

Dynamics on networks **83**
11 Introduction 83
11.1 Motivation 84
11.2 Research questions: graph dynamical systems 87
12 Dynamical network models classification 87
12.1 Techniques to describe dynamical networks models 89
12.2 Heterogeneous Opinion Status (HOpS) model setup 90
13 Results for Heterogeneous Opinion Status (HOpS) model 93
13.1 Analytic results for the HOpS model: dynamics on symmetric networks 93
13.2 The HOpS model dynamics on linear networks 94
13.3 The HOpS model on star-like networks 99
13.4 Numerical results for the HOpS model on random networks . 106
14 Conclusions and further directions 108

Conclusions **113**
15 Summary of all chapters 113
15.1 Evolving networks: methods of analysis for random models and data structures 113
15.2 Theoretical analysis of correlation networks 114
15.3 Dynamics on networks 114
16 Outlook 115

List of publications **117**

List of frequently used mathematical symbols **118**

Acknowledgements **120**

Literatur **125**

List of Figures

1 **Mindmap of connections between the chapters of the thesis** can be visualised as mathematical mushrooms. Idea is inspired by [8]. 21

2 Networks help to present the data from any system even though one should be careful with "fitting" the model as any structure. 22

3 **Solution of the Steiner network problem for three nodes A, B, C embedded in the Euclidean space**: the Steiner minimal network fulfills the optimality criteria when the edges AS, BS, CS form the angles 120 degrees from the Steiner point S. 23

4 **An example of an evolving *network growth model*** [69]: for each value of α, a model parameter, a network is being constructed, so that a new vertex with one edge is added to already existing nodes depending on ages of existing nodes. 24

5 **Example of the neural network evolving during life of C.Elegans.** The model of a neural network of C. elegans 3D was produced by VirtualWorm project Blender2NeuroML. 28

6 **Complex network represents a complex system** (a); an undirected network with only a single type of vertex and a single type of edge (b); an adjacency matrix, corresponding to this network (c) . . 29

7 **Schematic illustration of networks' measures:** degree, closeness, betweenness, brokearage. 29

8 **A network, represented as a multilayer network:** a network itself is in an upper layer, a degree field is in the bottom layer. Each node has degree value assigned with a color scheme: low degree values in bright green and high degree values in dark green. 30

9 **Degree distribution of a ER random graph** is calculated for a single random graph $G_{N,p}$ with $N = 10000$ nodes and a connection probability of $p = 0.0015$ for X_k, a number of nodes with degree k. The degree distribution has small deviations from the Poisson distribution $\frac{(Np)^k e^{-Np}}{k!}$. 32

10 **Different types of networks**: a. An undirected network with only a single type of vertex and a single type of edge; b. A network in which a each vertex and edge belongs to a certain type; c. A network with varying vertex and edge weights; d. A directed network in which each edge has a direction [191]. 34

11 **Example of ensemble of *evolving networks*** $\{G_1, G_2, G_3\}$ for changing set of nodes and edges. Evolving networks ensemble is represented as a multilayer network. 35
12 **The illustration of the time-series vectors associated with each grid box of the surface** (a). Correlation network is constructed based on the following rule: if the correlation value $C(T_i, T_j)$ between time-series T_i and T_j for the pair of grid boxes i, j is higher than a fixed threshold, then the nodes i, j are connected by a link (red line), otherwise there would be no link between the nodes. A correlation between time-series T_i and T_k is below threshold (red dashed line). Time-series vectors in the corresponding vector space are shown in (b). 37
13 **Flow-networks are constructed for two different flow width parameters**: $c = 200$ (a) and $c = 2000$ (b). 39
14 **Schematic illustration of various input variance factors in the START model of the Asian monsoon** (a-b), correlation networks reconstructed on grid (c-d), shown only 20% strongest links [213]. . . 40
15 **Degree fields for 40 yearly climate networks constructed for temperature time-series for (1970-2010). Illustration for *degree variability* measure between various degree fields.** All climate networks are constructed for the same link density of 5%. . 41
16 **Two networks can be isomorphic having different topological realisations on the plane**. It is important to mention that the definition of the *common component* of networks, embedded in the plane, depends on the numeration chosen for nodes. 42
17 **Estimation of variability of random networks using common component evolution function (CCEF).** CCEF for indexed set of random Erdős-Renyi networks with 100 nodes and different linking probabilities p marked by different colors. The dashed lines correspond to the analytical CCEF levels, the errorbars give the 1σ standard deviation for each index lag. 47
18 **Estimation of variability of parametrized flow-networks using common component function (CCEF)**. CCEF for the indexed flow-networks with increasing flow-width parameter c, and for different threshold values of adjacency matrices of the flow-networks. 48
19 **Estimation of variability of correlation networks from the START model using common component evolution function (CCEF)**. CCEF for networks from the START model with periodic (green) and linearly increasing (blue) forcing parameter F. The error bars indicate the standard deviation of the CC size estimates. 49
20 **Estimation of variability of climate networks using common component function (CCF).** CCEF of annual climate networks for the time period 1970-2011 C.E., error bars are presented as CCEF standard deviation of the respective time lag in years. 50

21 **Common-links-recurrence-diagram (a) and correlation matrix of the common link evolution (b).** Each point (i, j) in the diagram (b) corresponds to the value of the correlation coefficient $corr(i, j)$ between the common component functions $CCF(G_i, G_k)$ and $CCF(G_j, G_k)$. Lines with low values (marked at the bottom with arrows) are observable around strong ENSO years. 51

22 **Evolving average path length (A) and transitivity (B) network measures are calculated for each neighboring pair of yearly climate networks constructed for the period (1970-2010).** Most of the peaks of EAPL correspond to big El Nino (EN) years (red vertical bars), troughs of ET correspond to La Nina events (blue vertical bars) where color intensity of the bar corresponds to EN event strength. Data has spatial resolution $2.5° \times 2.5°$ covering the area between 2.5°S to 42.5°N and 57.5°E to 122.5°, i.e. 468 nodes. . 52

23 **The variability of degree fields for climate networks, constructed for each year in the period (1970-2010)** (a). The degree measure for the network constructed for correlations between degree variability in each node (b). Climate networks are constructed for the period (1970-2010). 54

24 **The Gulf stream current in the Atlantic ocean has a complex turbulent structure. The meandering flow is periodically changing in time** [38]. The image source is www.tidetech.org. . . . 58

25 **The interpretation of flow-networks concept:** nodes are parts of croase-grained continuous system, links imply information or heat exchange between nodes. Then network topological features identify characteristics of the flow systems, for instance, mixing in the flows. 61

26 **Orbits of Kuznetsova flow**, defined by Eq. (12) from [52], for parameter values $h = 1, \mu = 0, \omega = 1$ (a), $h = 1, \mu = 0.01, \omega = 1$ (b). 62

27 **The streamfunction for the velocity field of the meandering flow is constructed for** $t = 0$, subplot (a). It describes a jet flowing from left to right, more intense in the central meandering core. The streamfunction is plotted here for $\nu = 0$, and it is the same as for any other value of ν if $t = 0$ or a multiple of the flow period. The meander amplitude is changing in time as: $B(t) = B_0 + \epsilon cos(\omega t + \theta)$ for fixed parameters $B_0 = 1.2$, $c = 0$, $\omega = 0.4$, $\epsilon = 0$, $\theta = \pi/2$. The bounding streamlines for a time-dependent meander for the parameters values on the subplot (b). 70

28 **Illustration for *Example 1***: on a subplot (a) a node i has degree 3, the highest degree in the network. At the same time it is connected to nodes k_1, k_2, k_3, which are disconnected from each other. On a subplot (b) node i has degree 3, in addition, nodes k_1, k_2, k_3 are connected to each other, so that node i has high degree and also high clustering coefficient. 73

29 **Non-isomorphic network examples for the same degree sequence**. Illustration for *Example 2* in Subsection 8.4. 73

30 **Node degree centrality for the correlation networks constructed for different flows and decay rates**. The direction x is horizontal and y is the vertical. Panels A and B display the case of the static flow, $\nu = 0$. C and D are for the amplitude-changing case, $\nu = 0.7$. The dynamic network in this case is plotted at a time after $t = 0$ multiple of the flow period. Then, for all panels the streamfunction at the time plotted is the one shown in Fig. 27. Panels A and C are for the fast decay case $b = 1$, and B and D are for the slow decay, $b = 0.05$, of the transported substance. Other parameters as stated in the text. . . . 75

31 **Node clustering coefficient for the correlation networks constructed for various flows and decay rates**. Panels are for the same parameters as in Fig. 30. 76

32 **Degree distributions** (non-normalized) for the flow-networks for static meander (a) and moving meander (b) for damping parameter $b = 1$. Link density is 0.0035%, other parameters are defined in Subsection 8.2. Degree values are on the vertical axis, nodes' rank is on the horizontal axis. 77

33 **Global modularity measure for flow-networks constructed for various values of parameter ν,** which defines the mixing regime. Value of ν_c corresponds to transition between two various mixing flow regimes. 78

34 **The Lagrangian network is constructed using advective properties of flows**. The adjacency matrix is estimated using so-called Ulam's method [88, 228]. 79

35 **Illustrative example of a dynamical network model**: nodes represent humans (or separate communities). Links between nodes correspond to connections between humans, nodes size represents status of a human (HOpS model). Links and states of the nodes can evolve in time. 84

36 **Evolving twitter-network** at time step t (a), and at time step $t+m$ (b), when several edges are added in the network. 86

37 **Connections between people** have been topic of discussions besides artists and scientists since the ancient times. The fragment from Botticelli's painting (a) illustrates a human interaction (b). 87

38 **Illustration of dynamical network model with different types of equilibrium states**: total consensus (on the left) or separation into smaller groups (on the right), and in each of them consensus is reached. 88

39 **Properties of dynamical networks models are divided into several classes** depending on characteristics of functions, acting on node states, type of coevolution between nodes states and network topology. 89

40 **Representation of the group of people with two types of opinions** marked in black and white colors (a). Heterogeneous Opinion Status (HOpS) model representation of the group from three people (b), where the node status is represented by the node size. 91

41 **Illustration of the time-step of the HOpS model**: firstly, active node i and its neighbor j are randomly chosen; secondly, a state of an active node, opinion $op_i(t)$, is changed with probability p depending on the status difference for nodes i and j. A node status is encoded by a node size. 92

42 **Star-like networks**: simple star $k = 5$ (a), complex star (b). Hierarchical networks of different types: symmetric tree (c) tree with additional links between different hierarchal layers (d). 93

43 **The HOpS model on a linear network** for $L = 5$ nodes. A position of a border $x(t)$ between black and white nodes ($0 < x(t) < L$) is considered as a random walker. The special initial condition: node statuses decrease linearly, nodes with different opinions are separated by a border. The HOpS model, starting from such initial condition, reaches one of two final stable states: all nodes have the same color. 95

44 **Dynamics of the HOpS model on a linear network.** $x_0 \in [0, 100]$ (horisontal axis) is initial number of black nodes and $a = 0.5 \tanh \sigma\Delta_{st} + 0.5$ (vertical axis). Colorbar corresponds to the probability $p(x_0, a)$ for the system to come to one certain equilibrium, when all nodes are black, starting from x_0. 97

45 **Schematic diagram of a part of a full phase space of the HOpS model on a linear network is presented as a sequence of states and transitions between them.** Here the phase space is shown only for the HOpS model with a special initial state: $L - x(0)$ nodes from the left border are black, and $x(0)$ nodes from the right border are white at time step $t = 0$. 98

46 **Spectrum of a transformation matrix for the HOpS model on a linear network** for $L = 50$. Larger spectral gaps are observed for larger parameter values $a = 0.5 \tanh \sigma\Delta_{st} + 0.5$ values, as the result, this gives faster convergence towards the equilibrium. Spectra for $a = 0.5$ is blue dashed line, for $a = 0.7$ - red dashed line, for $a = 0.9$ - green dashed line. 99

47 **Equal states of the HOpS model on a *simple star*** with $k = 3$ leaves: the states are equivalent iff in both states the central node has the same color and the number of nodes-leaves with identical color is the same. 100

48 **Phase space of the HOpS model on a *simple star*** with $k = 3$ leaves (top), arbitrary number of leaves (bottom). The arrows between each two states of the HOpS model represent transition probabilities, which equal a and b correspondingly, Subsection 13.1. 105

49 ***The waiting time* for the HOpS model on random Erdos-Renyi networks depends on the probability of link creation** p, subplot (a). Random realisations of the model are made for fixed parameters: $N = 90$ nodes, $T_{max} = 100000$, $\sigma = 1$, with random statuses distribution for each realisation. The waiting time (vertical axis) is calculated for each parameter p (horisontal axis). *Log-log* plot for the standard deviation/mean values of the waiting time, subplot (b), where a peak for parameter of random ER network $p \approx 0.15$ indicates the transition. 108

50 **Change of nodes states in the HOpS model on a tree-like network** can be presented as "random walkers", traveling on the network: at each time step one node has a probability to change its state. 109

List of Tables

1 **Types of functional networks** 38
2 **Methods to characterize the networks evolution** 43
3 **Types of flow-networks** 79
4 **Algorithm for the time step of the HOpS model** 91

Introduction.

Bereshit bara Elohim et hashamayim ve'et ha'arets

Genesis

1 Research questions for the network theory and applications

Mathematics and physics allow us to understand and describe harmony of the world. Understanding goes much beyound a formulation of classifications and equations, which describes physical and mathematical structures. Any phenomena in the natural science of different types independently from the scales from the birth of new galaxies to the behavior elementary particles, can be described using the same general theoretical framework. The challenge is that most of the systems are incredibly complex.

In the thesis I develop a set of tools to approach problems of complex systems using statistical physics and algebraic methods, graph and probability theories. Such theories adequately describe systems composed of many interacting components when the precise structure of those systems' components can be neglected to some extent. Such problems of so-called "organized and disorganized" complexity have been commonly considered in statistical physics [130, 135]. The concept of complex networks is broad and general. In principle, any complex system can be considered as a network, consisting of interacting parts or elements, related to each other. For instance, recording measurements in data points of some continuous system, we obtain a discrete system, a coarse-grained analogue of a continuous system, with optional connections between data points. I discuss such connections in more details in Chapters II and III. From one side, networks allow effectively to visualise the main interconnections within a continuous system and to investigate a level of system's complexity, Fig. 2. While from another rather critical side the question arises, whether such system representation gives us any novel insights. I am going to come back to this issue especially in Chapters III and IV.

The thesis chapters are assigned with *the following questions*:

- *How do complex networks architectures emerge and how do networks evolve? In particular, what are characteristic spatial patterns of evolving functional networks?*
  Chapter II is dedicated to development and improvement of techniques to

analyze evolving networks. Another challenge is to distinguish between global and local aspects of complex networks reconstructed from a given evolving physical structure. To give an example, the Earth is an extremely complex evolving system [151], observations from the Earth system are typically rather short, noisy and far from being stationary [46, 60]. In addition, methods to characterize regime transitions in the climate as part of the Earth system always have been lacking. A representation of a given physical system by functional networks is one possible methodology. Therefore questions on evolving functional networks are definitely the ones, from which the Earth science could definitely benefit.

- *What is a direct physical meaning of functional networks, particularly, correlation networks? Can we find a relationship between topological features of the functional networks constructed for a given system and properties of the underlying flow system?*
  Chapter III is dedicated to the theoretical background of functional networks. Empirical orthogonal functions analysis (or EOF) are known techniques to study modes in climate system [184]. EOFs are eigenfunctions for so-called *correlation matrices* constructed from system's time-series. *Correlation networks* are defined by their adjacency matrices which is taken to be correlation matrices. This arises the corresponding questions: what is an interpretation of such correlation networks? Do network measures for correlation networks give new insights about the system's dynamics?

- A motivation to study dynamical processes on networks arises from questions of diffusion or spreading *on* networks: *How to characterize processes on complex networks? How microscopic and macroscopic properties of dynamical models on networks are interconnected?* This is a starting point for theoretical problems investigated in Chapter IV.

All in all, my thesis is concerned with some conceptual issues of networks of dynamical systems and strives to look at the interpretation of correlation and adaptive networks from a new angle. In Fig. 1 I schematically illustrate connections between the chapters: Chapter I "Introduction", Chapter II "Evolving networks: methods of analysis for random models and data structures", Chapter III "Theoretical foundation of correlation networks", Chapter IV "Dynamics on networks", Chapter V "Conclusions".

1.1 Evolving networks: spatial and temporal properties

> Then do not squander time; for that's the stuff life is made of.
>
> Benjamin Franklin

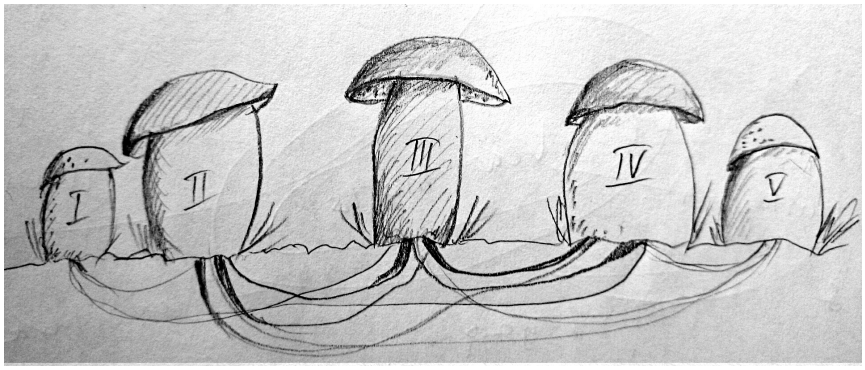

Figure 1: Mindmap of connections between the chapters of the thesis can be visualised as mathematical mushrooms. Idea is inspired by [8].

Historical part on graph theory

In this subsection I give the brief history of some graph theoretical problems in order to introduce my main research questions of Chapter II. Problems of the graph theory are connected to many different branches of mathematics [109]. Problems formulated in the language of complex networks which may be embedded or non-embedded in space, temporal or static, always have been providing an effective way of a visual representation of an abstract mathematical problem or analysis of a physical system. Who did not hear about the famous problem, suggested by Euler to find, "how not to pass through any of the same Konigsberg bridges"? Many problems of the graph theory are still remain unanswered, in particular, questions related to the graph evolution and graph isomorphisms [15], which recently received considerable attention in theory of computing because of connections with P-NP theory. Note also that a notion "network" here is used as the synonym to the mathematical object "graph".

Big class of problems in the graph theory is connected to *extreme properties of graphs* [34], e.g. finding the shortest path, or a graph with minimal cummulative length of edges, constructed on a given set of nodes, also known as Steiner problem. The Steiner tree is one of the classical (NP-complete) problems intensively studied nowadays [127]. In fact, finding a sub-graph that optimizes a global cost function is quite vital problem for applications for biological networks reconstruction, transportation science [54]. The Steiner minimal trees theory for small number of nodes in a given metric space is not that trivial problem. Geometric proof of the Steiner problem for three nodes is shown in Fig. 3: the edges of a minimal tree should connect in a central node (so-called Fermat point) forming angles of not less than 120 degrees. There are many other solutions to this problem, even formulated in physical terms [237]. In the XX century Kuratowski linked the graph theoretical problems to the topological problems with his new at that time idea "that planarity was nothing but topology", he proved the planarity theorem on a mathematical forbidden graph characterization of planar graphs. Another bright example of the graph theoretical problem is for instance Five Color Problem [109], which was followed up by many generations of

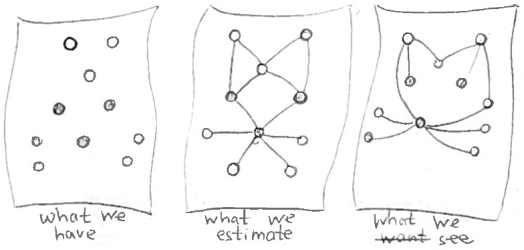

Figure 2: Networks help to present the data from any system even though one should be careful with "fitting" the model as any structure.

mathematicians and made a big progress recently [144, 210]. Already from the graph theoretical problems listed above one can see, that graph theory is a field of an active research with large number of analytical and numerical problems.

Applications of the graph theory have been considered in many fields where one network or a network ensemble represents a dynamical system. Hence it is essential to develop analytical techniques to study *networks evolving in time*. One of the problem is particularly to investigate ensembles of random networks [77, 78]. An example of evolving network growth models is shown in Fig. 4, where at each time step a new edge is added following a stochastic rule. General form of this rule can be given by: at each time step a new edge is added between node v_i with an age Ag_i and node n_j with an age Ag_j with probability $p_{ij} = f(Ag_i, Ag_j, \alpha)$, where α is some parameter of growth. In [68] the reparticular example of growth model is considered, where the aging is proportional to $\tau - \alpha$, where τ is the age of a vertex. This network grows clockwise starting from a vertex below on the left and at each time step, a new vertex with one edge is added. Are there any "simple" analytic ways to characterize evolving networks? In the following subsection I introduce problems on evolving networks, the main focus of Chapter II.

Evolving network theory

Temporal or evolving networks are natural continuation of the static networks. In *Chapter II* I address the questions on characterisation of a similarity between networks in the global and local scales. The reason of increasing interest to study temporal networks comes from the fact describing how a graph is wired, helps us understand, predict and optimize behavior of dynamical systems [122]. One of the existing approaches for comparing network local structure is based on the significance profile of small subgraphs in the network compared to randomized networks [175]. Generally speaking, characterizing and understanding the structure and the evolution of spatial networks is thus crucial for many different fields ranging from urbanism to epidemiology.

I approach the problem to quantify changes of network structures in numerated ensemble of networks as follows. I evaluate an overlap of sets of edges for each pair of

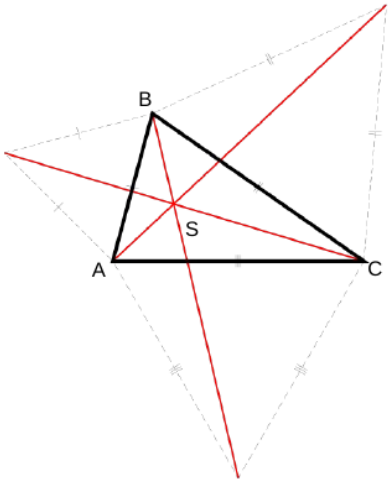

Figure 3: Solution of the Steiner network problem for three nodes A, B, C embedded in the Euclidean space: the Steiner minimal network fulfills the optimality criteria when the edges AS, BS, CS form the angles 120 degrees from the Steiner point S.

i and $i+1$ networks (*the common component* of networks). This allows to introduce a discrete-time function (*the common component evolution function*) depending on a number of enumerated network in a network ensemble. To test suggested approach I first calculate *the common component evolution function* to well known random networks ensembles. *The common component evolution function* properties are discussed in details in Chapter II. For which systems this would be an evolution of networks would be an actual issue to consider? Recently analysis of climate time-series was applied to analyze climate variability [62, 200, 203]. In fact, climate networks can be considered as an enumerated ensemble of networks constructed to neighboring time periods. In an addition to existing methodologies to detect tipping points and phase transitions in a climate system, the common component evolution function seems to be a helpful tool. For which exactly climate systems an estimation of variability and tipping points is an essential question? One such system is the Indian monsoon phenomenon, understanding its variability of and its interaction with ENSO [145, 259] remains one of the most vital questions in climatology. Overall, the common component method of temporal networks characterisation can be used to perform general analysis for a broad class of continuous and discrete systems which can be associated with an enumerated ensemble of networks.

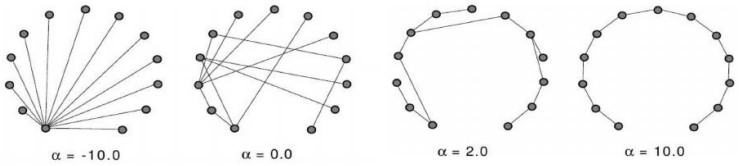

Figure 4: An example of an evolving *network growth model* [69]: for each value of α, a model parameter, a network is being constructed, so that a new vertex with one edge is added to already existing nodes depending on ages of existing nodes.

1.2 Theoretical analysis of correlation networks from the advection-diffusion dynamics

Led tronulsa - slyshny kriki sredi jasnogo dnja...

A.Chekhov

Chapter III is dedicated to the theoretical analysis of correlation network measures constructed from various flow systems. As has been mentioned in Subsection 1.1, network techniques were applied in many fields: social systems, fluid dynamics, climate, multivariate time series analysis [63, 113, 199, 224]. Climate is a very large system with a vast number of parameters and influences, some of which, like advection and diffusion, can be modeled by differential equations. Then the question arises: How do correlation networks for advection-diffusion dynamics look like, what are their prominent topological properties? These questions form main motivation of Chapter III, where I try to gain a deeper understanding of a complex interplay between dynamical systems and corresponding complex networks, more specifically, so-called *flow-networks*. Flow-networks were recently introduced using various frameworks of flow systems [179, 215, 254]. First of all, flow-networks representing the advection-diffusion dynamics are an interesting mathematical object for investigation, independently from any particular particular interest, due to their remarkable topological features. Second, flow-networks are related to problems of finding analytical or numerical solutions for a system with time-independent and time-dependent underlying velocity fields, which play an important role for physical applications. Third, the flow-networks construction method provides promising tools to analyze mixing properties of fluids [198, 217].
To summarize, methodology for the flow-network representation of dynamical systems is of potential interest for a broad audience within the physics community. Moreover, the flow-networks method may be further adopted in various applied fields, where questions on any spatially extended dynamical system are valid.

1.3 Dynamics on networks

On v drugo raz zakinul nevod...

A. Pushkin

Diffusion-like processes on various structures, such as grids, trees, is a very alive topic [53, 157, 238]. In Chapter III I define a new method to analyze diffusion-advection processes using discretisation on grids, while in Chapter IV I consider spreading processes on different types of networks. At the first place, the "real-world" motivation arises from epidemiological studies virus spreading [97, 152] or spread of information in the webgraph [121, 276]. At the same time, the real-world complex biological and social systems [44] are easily translated to the framework of adaptive networks, or more generally of *"dynamical network models"* [222]. In Chapter IV I inquire, how analytically we can characterize diffusion-like process on dynamical network model? Looking ahead, I define a particular type of dynamical network models, the heterogeneous opinion-status network model. It is known, that prototypical dynamical models help to identify properties general system's properties [32, 243]. In general, conceptual models, like the one presented in Chapter IV, expose an interrelation between specific questions on dynamical network models and a generic class of problems from statistical physics.

Evolving networks: methods of analysis for random models and data structures

2 Introduction: static and evolving networks

> Our imagination flies - we are its shadow on the Earth.
>
> Vladimir Nabokov

Networks have been extensively used to study phenomena in sociology, engineering and natural systems [21, 44, 132, 170]. Various aspects of *temporally changing networks* have been considered for sociological and biological networks. In [1] a random network growth and evolution in response to addition or rewiring of links between nodes was analysed. It has been found that a graph topology changed depending on a frequency of link changes. In [89] function of graph changes was tracked using a stochastic block model for evolving networks to investigate evolutionary effects in email networks and gene regulation. Ubiquitous examples of *evolving networks* in nature include networks of citations of scientific papers with references as links [124], social networks of contacts, sexual contacts [152], net of WorldWideWeb and many others. For example, neurobiologists were able to visualize a graph of a neural network of a worm C.Elegans, which is known as one of the simplest organisms with a nervous system from approximately 302 neurons [233]. A network of a biological system is obviously changing during life of a worm, a network at one certain time period is shown in Fig. 5. Another representative example is a biological metabolic network [41], a graph of interactions forming a part of a energy generation. In a small building block synthesis metabolism vertices represent substrates and products, and edges represent interactions. Studying such *functional networks* one can deepen our understanding of causality of processes of an analysed complex system.

The *main topic of this chapter* is a problem of characterisation of evolving networks which are many-parametric objects to describe: how to derive a function, representing networks evolution? Firstly, I give an overview over general network definitions, real world examples of evolving and random networks. In the methodological Section 3 I present new network indices, validating them on random network ensembles. Then in Subsection 3.2 I introduce evolving network measures to analyse and interpret spatial and temporal aspects of networks evolution. I demonstrate presented techniques for networks embedded and non-embedded in space. These techniques are then applied to networks constructed from various data sources.

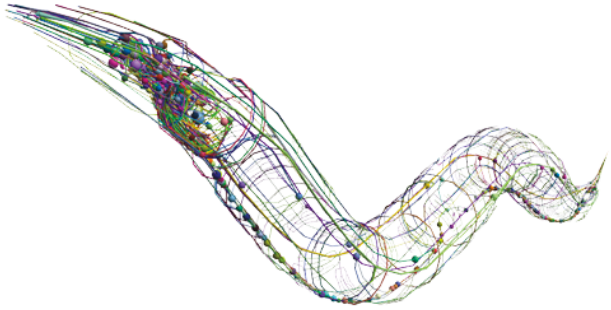

Figure 5: Example of the neural network evolving during life of C.Elegans. The model of a neural network of C. elegans 3D was produced by VirtualWorm project Blender2NeuroML.

2.1 General definitions from network theory

The study of networks is one of the most developed branches of discrete mathematics. It is important first to give definitions of *static networks* measures and network attributes in order to introduce them later for more general network concepts like *evolving networks*.

Definition. A *network* (or graph) G is a set of items, called vertices or nodes, with connections between them, called edges, Fig.6(b). Formally, a *network* (or graph) G can be considered as a pair $G = (V, E)$, consisting of a finite set $V = v_1, ..., v_N$ of vertices (or nodes) and a finite set $E = e_1, ..., e_M$ of edges (or links), where each edge $e_i = (v_k, v_l)$ is a two-element subset of set V of connected nodes v_k, v_l.

Let us consider a square $N \times N$ matrix A with an element $a_{ij} = 1$ if the nodes v_i and v_j are connected and $a_{ij} = 0$ otherwise. Such a matrix is called *adjacency matrix*, Fig. 6(c). One can represent a complex system as a network, Fig. 6(a,b) when the system has an explicit graph structure. Descriptive real-world examples of networks are the Internet, social networks of connections between individuals and many others. The terms "network" and "graph" are used synonymously here.

2.2 Network measures and characteristics

> I tell you the truth, when you did it to one of the least of these my brothers and sisters, you were doing it to me
>
> Matt. 25:34-40

One of the most efficient ways to tackle the network structure is to use network measures. A network can be considered from different prospectives as: local or global structure. Depending on a type of the approach used, one gets a local network measure (a vector of values for each node) or a global one (one characteristic value

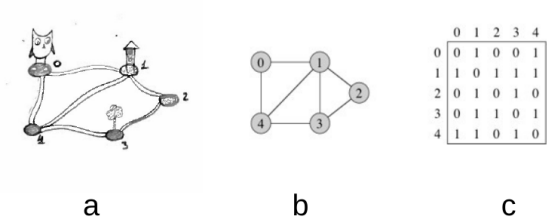

Figure 6: Complex network represents a complex system (a); an undirected network with only a single type of vertex and a single type of edge (b); an adjacency matrix, corresponding to this network (c)

for the whole network).
Below I give definitions of network measures in order further to generalize and to test them on different network types: randomly generated networks and correlation networks constructed from data, Section 4.
Degree centrality:
denoted by deg_i is the number of edges connected to a vertex i, Fig.7. Note that the degree is not necessarily equal to the number of vertices adjacent to a vertex, since there may be more than one edge between any two vertices. A directed graph has both an in-degree and an out-degree for each vertex, which are the numbers of in-coming and out-going edges respectively. For weighted networks *weighted degree centrality* deg_i^w for node i is defined as $deg_i^w = \sum_j w_{ij}$, where w_{ij} is the weight of links incident to node j. *Degree field* is the sequence of degree values for all nodes of the network.

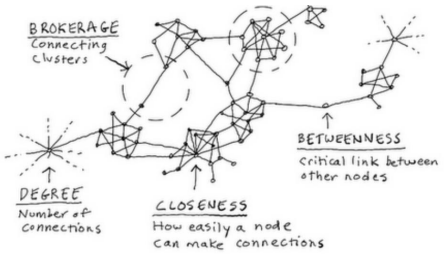

Figure 7: Schematic illustration of networks' measures: degree, closeness, betweenness, brokearage.

Clustering coefficient:
is a measure of the degree to which nodes in a graph tend to cluster together. The

local cross-clustering coefficient C_v^{ij} is defined as:

$$C_v^{ij} = \frac{|\Gamma_v|_E}{\frac{k_v}{2}}, \tag{1}$$

where $|\Gamma_v|_E$ denotes the number of edges in maximal subgraph spanned by the neighborhood Γ_v. Then C_v^{ij} is the probability that two randomly drawn neighbors of vertex v from subnetwork G_j are neighbors themselves, where v belongs to subnetwork G_i. The global clustering coefficient for the whole network is the number of closed triplets over the total number of triplets (both open and closed), Fig.8. Evidence suggests that in most real-world networks nodes tend to create groups characterised by a relatively high density of links between nodes. Some algorithms for calculation of a clustering coefficient are using random walk theory [270]. An interesting comparison can be made for the degree and clustering measures. More detailed information on this is given in Chapter III.

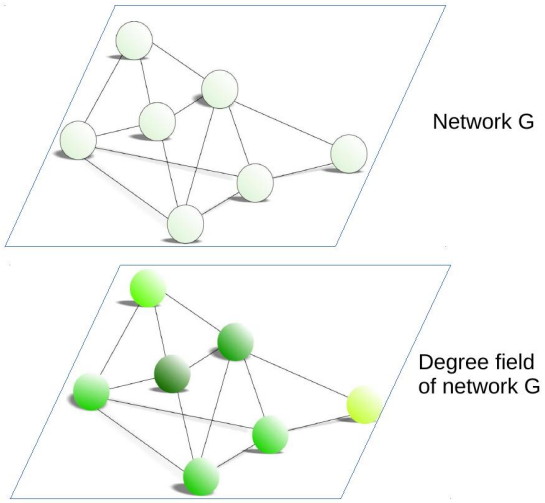

Figure 8: A network, represented as a multilayer network: a network itself is in an upper layer, a degree field is in the bottom layer. Each node has degree value assigned with a color scheme: low degree values in bright green and high degree values in dark green.

Betweenness centrality:
is a measure of a node's centrality in a network. It is equal to the number of shortest paths from all vertices to all others that pass through that node. Betweenness centrality is a more useful measure, than just connectivity, of both the load and importance of a node. The former is more global to a network, whereas the latter is only a local effect. Suppose that σ_{sk}^i is the number of geodesic paths from vertex s to vertex k that pass through i and σ_{sk} is the number of all paths from vertex s to

vertex k. The formula for betweenness centrality in an unweighted case for node i is:

$$g(i) = \sum_{s \neq i \neq k} \sigma^i_{sk} / \sigma_{sk} \tag{2}$$

This measure can also be generalized for the weighted networks. Some algorithms for calculating betweenness are described in [83].

Random walk betweenness centrality:
is another *betweenness centrality measure*, which includes the contributions from essentially all paths between nodes, not just the shortest. Since it is computationally expensive to consider all the shortest paths and may be more realistic to consider a random walk betweenness centrality. Especially, when the shortest-path-condition does not hold for a considered system, there might be other measures of betweenness of a vertex. A more recent version of this measure is proposed in [188].

Closeness centrality:
C^C based on graph-distance, proposed by [55], defined as an inverse of an average distance from all other nodes. The closeness centrality C^C of every vertex v_i is defined as:

$$C(i)^C = \sum_{j \in \nu}^{N} 2^{-d_{ij}},$$

where d_{ij} is the shortest distance between vertices i and j on graph G $C(i)^C$ can practically be read as a time until arrival information spreading in a network.

Empirical Orthogonal Functions (EOF) measures or Eigenvector and information centrality. Eigenvalues of the network adjacency matrix are describing the patterns of the variability. The eigenvectors of so-called correlation matrix constructed from the time series are used for data analysis [227, 245]. The largest variability of the time series can be shown by first EOFs of the correlation matrix and therefore, EOFs are widely used in climatology. This measure is also called *eigenvector-centrality*, it shows the influence or importance of the node in the network. To compute the eigenvalues for correlation matrices for the time series from data with high resolution may be computationally expansive therefore special methods for data analysis are developed. So called "information centrality" is used for analysis of social network analysis [118] and is connected to the eigenvector centrality measure.

Assortativity measure:
is a preference for network nodes to attach to others that are similar in some way, therefore this measure estimates "mixing" in network, i.e.how well nodes with different degrees are mixed. A formal definition for *assortativity* of node i is sum of degree values deg_j of all neighbor-nodes v_j: $\sum_{j:e_{ij} \in E} deg_j$. Often assortativity is examined in terms of a node's degree [191]. Correlations between nodes of similar degree are often found in mixing patterns of many observable networks. In social networks, nodes tend to be connected with other nodes with similar degree values. This tendency is referred to as assortative mixing, or assortativity. On the other hand, technological and biological networks typically show disassortative mixing, or dissortativity, as high

degree nodes tend to attach to low degree nodes [268]. Important to note, that this measure has a big potential of further generalisations for the degree sequences of each node i for evolving networks [253].

Anisotropy:
is defined as the averaged direction of links in the networks for fixed node. Thus it can be useful to apply to different kinds of transportation networks or flow-networks, introduced in work [147], as well as in Chapter III and [179, 215, 254].

Degree measure for networks embedded in space:
Let us consider a network embedded in space and each edge weighted by the edge distance w_{ij}. Then degree for the embedded in space network can be defined as $\sum_j w_{ij}$.

The following *network characteristics* are based on the network measures introduced above:

Degree distribution:
is denoted by p_k and defined to be the fraction of vertices in the network with degree k. Equivalently, p_k is the probability that a vertex chosen uniformly has degree k. A plot of p_k for any given network can be formed by making a histogram of nodes degrees. It has been found that many real-world networks have a power-law degree distribution i.e., the probability density function $p(k)$ (k is the degree) has the form: $p(k) = ak^{-\gamma}$. Networks with a power-law degree distribution, for this reason, are often called âscale-free networksâ, as the probability density function f obeys the relation: $f(ax) = bf(x), x \in R$. Degree can also be a building block in some complex schemes for analysing networks. Then degree distribution, Fig.32, can be considered as a proxy for âimportanceâ in constructing a method to quantify âhierarchical organisationâ in networks [2].

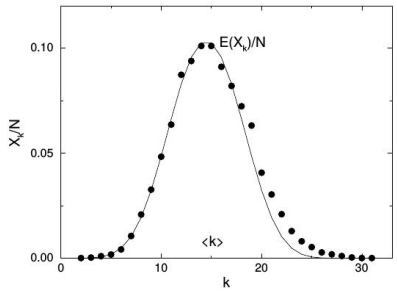

Figure 9: Degree distribution of a ER random graph is calculated for a single random graph $G_{N,p}$ with $N = 10000$ nodes and a connection probability of $p = 0.0015$ for X_k, a number of nodes with degree k. The degree distribution has small deviations from the Poisson distribution $\frac{(Np)^k e^{-Np}}{k!}$.

Link length distribution:
is a global network measure which can be introduced for weighted networks embed-

ded in space where a length of a link weight equals a metric length of a link. By definition link length distribution $l(j)$ is a fraction of links with the same link length. Therefore this measure combines topology of a network and a geometry of embedding. Complete link length distribution is a link length distribution of a complete graph of a set of spatially embedded nodes. Together with *degree distribution* link length distribution can be used to visualize how many short and long links are prevailing in a network. For an ensemble of networks one can use techniques to compare link length distributions for different networks. Often the distribution for real networks of the link lengths follows a power law [75].
There exists a variety of other network measures, such as bridging and bonging measures, inspired by the work of M.Granovetter, mixing network measures [189], symmetricity of network [123], structural equivalence measure [156]. Measures for non-static networks are, in general, based on static network measures [56, 265]. An important property of any linear centrality measure $c(i)$ for node i is that centrality measure should fulfill the condition $c(i) \leq c(j)$, if a set of surrounding nodes of node j includes a set of surrounding nodes of node i. Note that depending on a type of network one needs to use corresponding network measures. A basic classification of static and evolving networks, based on methods of their construction and representation, are discussed in Subsection 2.3.

2.3 Basic types of networks

A classification, based on properties of nodes and edges, starts with a simple network without any attributes:
a) Network $G = (V, E)$ on the set of nodes V and edges V without any additional properties is shown in Fig. 10(a), where a set of edges E_i which can be defined as a set of functions between set of nodes: $E : N \to N$.
b) A set of nodes V can be divided into different subsets, Fig. 10(b), corresponding to various nodes' attributes. In chapter IV this I turn back to this type of networks. Similarly, edges can be attributed with properties, such as weights, or belong to different groups.
c) When edges have additional properties, such as weights, the network is *weighted*, $G = (W, E)$, Fig. 10(c), where W is a set of nodes' weights. Unweighted networks can be considered as simple case of weighted networks, when all weights of links have the same weight.
d) Graphs with directed edges are called *directed graphs* or digraphs, $G = (V, E)$: $e_{ij} \in 0, 1, e_{ij} V e_{ji}$, as it is shown in Fig.10(d). An edge is directed, if it is linked in only one direction, and undirected if it is connected in both directions. Directed edges can be thought of as sporting arrows indicating their orientation.
Additional nodes' attributes, such as nodes numbers or nodes' positions, make a difference for nodes

For instance, nodes' enumeration plays an important role for the graph isomorphism problem [15]. Nodes' positions are characterizing, how a network is *embedded in space* X. Let is consider vertices $v_i \in V \in \{1, ...N\}$ of graph $G = G(V, E)$ belonging to

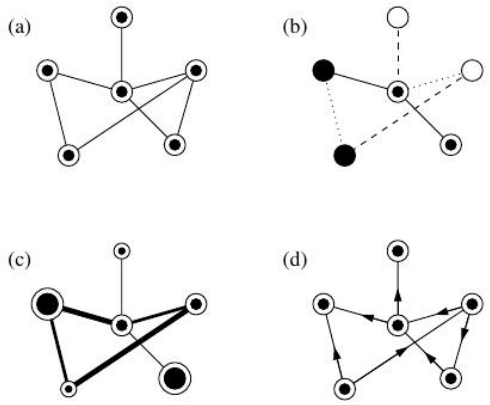

Figure 10: Different types of networks: a. An undirected network with only a single type of vertex and a single type of edge; b. A network in which a each vertex and edge belongs to a certain type; c. A network with varying vertex and edge weights; d. A directed network in which each edge has a direction [191].

n-dimensional space X space [75]. Then we associate each node v_i with its coordinates $(k_{ix_1}, ...k_{ix_n})$, where $K = \{k_{jx_1}, ...k_{jx_n}\} \forall j \in \{1, ...N\}$ is a set of coordinates of all nodes v_j. Being spatially embedded in a certain metric space (X, ρ) implies that an edge length between each two nodes is estimated using metrics space ρ: $|e_{ij}|_\rho$. In this chapter I consider static and evolving networks in Euclidean space. Basic classification of evolving networks is presented in the next subsection.

Classification of evolving networks

An evolving network is a at the next "level of complexity" in comparison to a static network. Before to introduce new global and local network measures for evolving networks Formally, I define evolving networks as follows.

Definition. Let us call *evolving networks* an ensemble of static networks or a sequence of networks $\{G_t, t \in [1, T]\}$, where $G_t = G(V_t, E_t)$ for each time step t, so that set $\{G_t, t \in [1, T]\}$ is linearly ordered in time.

In [176] a process of network evolution is encoded via the growth of set V and E, so called *network growth model*. A wide variety of of such network growth models were exposed in [26, 78]. In the thesis I examine evolving networks for fixed set of vertices and evolving in time set of edges, if not stated otherwise.

Formal evolution of networks in time can be viewed as a transformation by function F, acting on a network G_t for each time-step t such that: $F(G(V_t, E_t)) = G(V_t, F(E_t)) = G(V_t, E_{t+1})$. Function F acts on a set of edges for a fixed set of nodes, transforming of a set of edges for each time step t. This formal definition is discussed in details in Chapter IV. Note that temporal networks can be visualized as multilayer networks [205], where each layer corresponds to a separate network, shown in Fig. 11. Depending

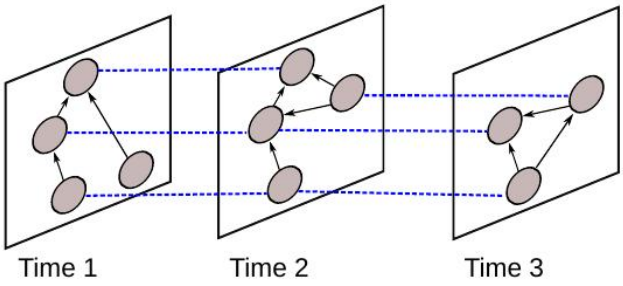

Figure 11: Example of ensemble of *evolving networks* $\{G_1, G_2, G_3\}$ for changing set of nodes and edges. Evolving networks ensemble is represented as a multilayer network.

on a research question one needs to use different representation of ensemble of evolving networks. A special type of networks' ensembles, **randomly generated networks**, is introduced below.

Randomly generated networks

In a series of seminal papers from the 1950s [77], one of the earliest theoretical models of random graphs were proposed and studied. Yet similar idea of analysis of random structures has been present in works on random matrix theory [11, 12, 266]. Here I give a brief overview of the main types of random networks, to some of them I specifically come back later on in Chapters III and IV:
1) One way to introduce the ***random Erdos and Renyi network model*** (ER), denoted as $G_{N,p}$, is that each possible edge between two vertices is present in a network with independent probability p, and absent with probability $1-p$. More formally speaking $G_{N,p}$ is an ensemble of graphs of n vertices in which each graph appears with a probability appropriate to its number of edges.
It is important to mention one interesting feature, which was demonstrated in [77]: ER model shows a phase transition with increasing average degree of a vertex at which a giant component forms. I come to this issue in Chapter IV.
2) Another class of random networks are ***growth networks constraints***, for instance, *Barabasi-Albert graph*. The algorithm of its generation can be described in terms of non-equilibrium networks [67]. It has been noticed that a random graph differs from any real-world network in some fundamental ways. This was noted in the recent literature [16, 242]. First, as pointed out by Watts and Strogatz [262] Erdos and Renyiâs model does not show strong clustering or network transitivity, while real-world networks do. The probabilities of vertex pairs being connected by edges are by definition independent, so that there is no greater probability of two vertices being connected if they have a mutual neighbor than if they do not. Second, their degree distributions, a point which has been emphasized particularly in the work [16], The probability p_k that a vertex in an Erdos-Renyi random graph has degree of k is given by the binomial distribution.

3) In the last years ***extensions of network growth models*** have been presented in [1, 192]. This type of random networks gives an arbitrary degree distribution with generalized power-law form. So called correlated networks concept is one of possible solutions to this problem.
Many interesting concepts in physics are connected with random networks, some of them are described in series of the recent works [107, 139, 196, 262]. Defining random networks embedded in space, one can introduce additional geometric properties for these networks [27, 190, 191]. Typical examples of it are geometric graphs, Apollonian networks [21], planar Erdos-Renyi and Barabasi-Albert networks obtained by the rule of rejecting links if they destroy planarity.

2.4 Functional networks

General networks classification goes beyond networks types listed in Subsection 2.3. *Functional networks* represent functional structure of a complex system. In this chapter and in Chapter III I am focusing on a specific type of functional networks, so-called correlation networks.

Correlation networks

Definition. A *correlation network* is a network defined by its adjacency matrix C of size $N \times N$, obtained from N time-series $T_i, i \in [1, N]$, which are associated with nodes $i, i \in [1, N]$. Correlation matrix C is constructed by the following rule: each component of matrix C_{ij} equals a correlation value $C(T_i, T_j)$ between time-series (or vector-states) T_i and T_j for each pair of grid boxes i, j (nodes of a correlation network). Furthermore, a correlation adjacency matrix can be thresholded: if a value $C(T_i, T_j)$ is higher than a fixed threshold, then nodes i, j are connected by a link, Fig. 12. Interestingly, a threshold graph [47] is related concept of such network construction.

Let us discuss a mathematical definition of a correlation network. Consider a fixed domain on Earth, which is cross-grained into N grid points with time-series attached to each grid-point. Let us call a vector of time-series for each grid box i for T time steps, a *state-vector* $T_i = (T_i(t), ...T_i(T))$, which belongs to a T-dimensional vector space H. An inner product between two state-vectors has symmetricity, linearity, positive definition properties. By definition of a correlation network, if an inner product (or a correlation value) between two time-series, associated with these grid points (network nodes), is above a certain threshold, then there exists an edge between two grid points, forming a set of edges E. Hence, a network with set of edges E represents time-series for a period of time $[1, T]$, and as the result, correlation network topology represents the underlying process. Now let us look more precisely into this interrelation between topology and dynamics: if time-series for each grid points i and $j, \forall i, j \in [1, N]$ are correlated, then state-vectors T_i and T_j are "close" in a vector space H, as it is shown in Fig. 12. One can say that this interrelation maps a set of time-series to a network.

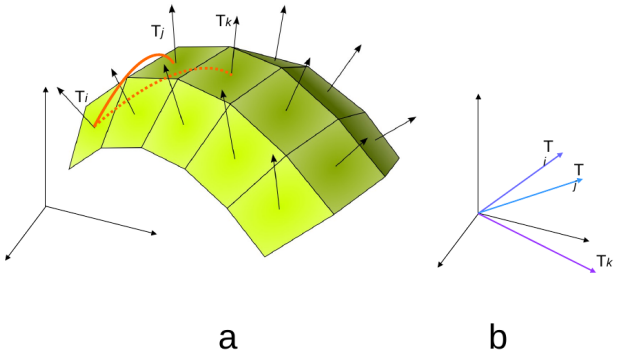

Figure 12: The illustration of the time-series vectors associated with each grid box of the surface (a). Correlation network is constructed based on the following rule: if the correlation value $C(T_i, T_j)$ between time-series T_i and T_j for the pair of grid boxes i, j is higher than a fixed threshold, then the nodes i, j are connected by a link (red line), otherwise there would be no link between the nodes. A correlation between time-series T_i and T_k is below threshold (red dashed line). Time-series vectors in the corresponding vector space are shown in (b).

Constructing correlation networks for k sequential time windows, denoted by $[1, T^1], [T^1 + 1, T^2], ...[T^{k-1} + 1, T^k]$, we get an ensemble of correlation networks evolving in time. Correlation between time-series for various nodes can be estimated, for instance by linear *Pearson correlation*, which is defined as a normalized covariance. The covariance between each pair of time-series vectors $T_i(t), T_j(t), i, j = 1, ..., N$ with zero mean is defined as a sum over T time steps of coeval values of time series:

$$\mathrm{Cov}(T_i, T_j) = \langle [(T_i - \langle T_i \rangle)(T_j - \langle T_j \rangle)] \rangle, \tag{3}$$

where $\langle T_i \rangle$ denotes the expected value of a vector T_i, also known as a mean value.
Proposition I. Every symmetric positive semi-definite matrix is a covariance matrix.
Proof. To see this, suppose M is a $N \times N$ positive-semidefinite matrix. From the finite-dimensional case of the spectral theorem, it follows that M has a nonnegative symmetric square root, that can be denoted by $M^{1/2}$. Let $\mathbf{X}$ be any $N \times 1$ column vector-valued random variable whose covariance matrix is the $N \times N$ identity matrix [91]. Then

$$\mathrm{var}(\mathbf{M}^{1/2}\mathbf{X}) = \mathbf{M}^{1/2}(\mathrm{var}(\mathbf{X}))\mathbf{M}^{1/2} = \mathbf{M}. \tag{4}$$

However, mapping between all possible combinations of state-vectors and all semi-definite matrices is a surjection [273], i.e. for several ensembles of time-series can have the same correlation matrix.
From this proposition follows that some information from time-series is "lost", since . However, a topology of a correlation network infers some characteristic features of time-series (for details see Chapter III).

Table 1: Types of functional networks

	Types of functional networks
(1)	**correlation networks**, for instance, climate networks, discussed in Chapters II and III [62, 249, 252];
(2)	**synchronisation networks**, i.e. networks constructed using event synchronisation method [30, 163, 208];
(3)	**recurrence networks**, based on neighborhood relations in phase space [64, 165]
(4)	**causal networks**, in particular, climate networks from time-series inferred causal strength [115, 216];
(5)	**transport flow-networks**, such that an adjacency matrix is a correlation matrix, calculated from a dynamical system, Chapter III, [147, 178, 254].

Functional networks can be classified depending on the type of observable and method of functional connectivity calculation (for instance, correlation using Spearman's rank coefficient, mutual information [80]), shown in Table 1.

Classification of functional networks

A classification of functional networks can rely on a type of data structures, for instance, paleoclimate networks are reconstructed from paleoclimate data [212], while neural networks correspond to time-series estimated from a brain activity [271].
Here I present a classification of functional networks, depending on the method of revealing a functional structure from a data. *Linear correlation* is most commonly quantified by (Pearsonâs) correlation coefficient. While non-linear connectivity measures are in use, for data with approximately Gaussian distribution linear correlation is practically sufficient. The correlation matrix can be subsequently transformed into an unweighted graph "binarized" by choosing a threshold and assigning links only to pairs of nodes with over-threshold correlation [57, 116].
Functional climate networks gave rise to many interdisciplinary questions. This has been a trigger to study correlation networks as a new theoretical mathematical object, linking correlation network structure and fluid flows dynamics. One needs to be careful with the interpretation of climate networks. Therefore I inquire, to which extent the network measures can be interpreted? I investigate simulations of spatially embedded processes using correlation networks, see Table 1: correlation networks (1) from the conceptual climate model **S**patio-**T**emporally **A**utoco**r**related **T**ime series model START, introduced in [213]; yearly climate networks generated for the temperature data for Indian subcontinent (1); networks analytically generated

from flow fields (5), as described in [178]. Construction and details about the climatic data are given below.

Generation of evolving flow-networks

The idea of the *flow-networks* is to construct the correlation networks directly from the dynamical system (without time-series calculation for each time step). It is possible to construct flow-networks from continuous or discrete approach. For the discrete flow-networks I refer to Chapter III, where this method is introduced. According to continuous approach, [178], flow-networks are constructed directly from a velocity field using a correlation measure based on the temperature profiles, which results from a temperature peak spreading. The idea behind this method is that certain velocity field (advective term) corresponds to network (I refer to Chapter III where the discrete flow-networks method is presented in more details). The velocity function considered here is parametrized by the parameter of flow width c is varied for the flow-width from 200 to 2000 in 10 steps, Fig. 13, thus gradually changing the flow network:

$$\vec{v}(x,y) = \begin{pmatrix} e^{\frac{-(y-0.5x)^2}{c}} \\ 0.5e^{\frac{-(y-0.5x)^2}{c}} \end{pmatrix} , \tag{5}$$

The positions and the number of nodes are kept constant. For each value of c I obtain a correlation matrices $C_1, ..., C_{10}$, and thresholding these matrices by different critical values I obtain set of adjacency matrices.

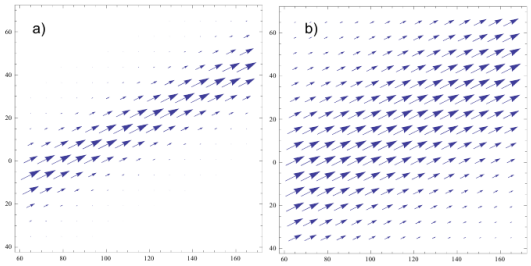

Figure 13: Flow-networks are constructed for two different flow width parameters: $c = 200$ (a) and $c = 2000$ (b).

Generation of ensemble of evolving correlation networks for conceptual START-model

The START model (**S**patio-**T**emporally **A**utoco**r**related **T**ime series model) is a conceptual climate model, designed in [179, 213] (for more details on conceptual climate models see Appendix). The model philosophy is to demonstrate the propagation of

climate variability through flows in a spatially extended domain. By implementation it is a statistical toy model with three independent spatial components that react differently to applied external forcing F, Fig. 14. Correlation networks generated from

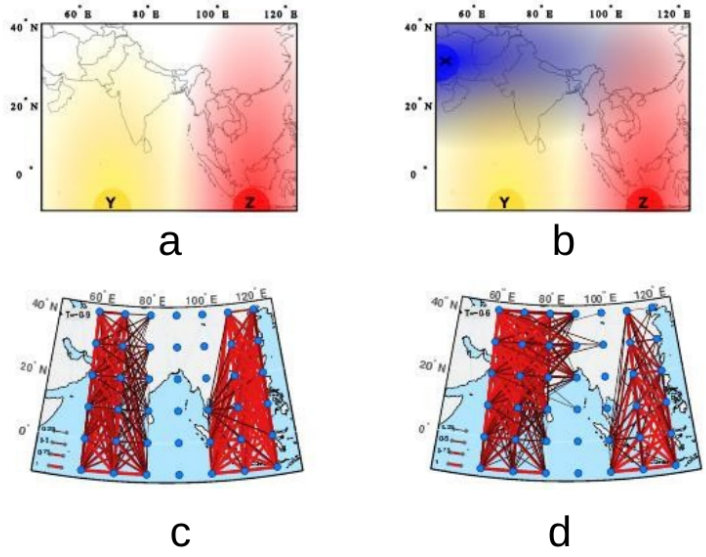

Figure 14: Schematic illustration of various input variance factors in the START model of the Asian monsoon (a-b), correlation networks reconstructed on grid (c-d), shown only 20% strongest links [213].

the time-series of START undergo a distinct transition when the forcing parameter F is changed: For $F = -1$ the network is partitioned into two vertical connected areas. For $F = 0$ horizontal cross-links have appeared and link the two sections. At maximal forcing, for $F = 1$, there is one large, horizontally oriented component. The transient simulations with a 6×7 sampling grid were performed [213], for 20000 time steps and 100 ensemble members each. In the first run the forcing parameter was increased linearly from the start to the end of the simulation. In the second run the forcing parameter $F(t) = \sin(t/2000)$ was periodically changed. Networks were constructed based on the 20% strongest links in the correlation matrices obtained for each 100-step-long time window. Due to the stochastic component, networks constructed for different ensemble members, but for the same time period may be quite different, networks for different periods of same ensemble member may be quite similar.

Generation of evolving networks from Asian monsoon data

Climate networks approach: analysis of variability of the complex systems. The climate system is an extraordinarily complex continuous system. As it was shown in [60] the variability of climate systems exhibits a huge variety of phenomenon,

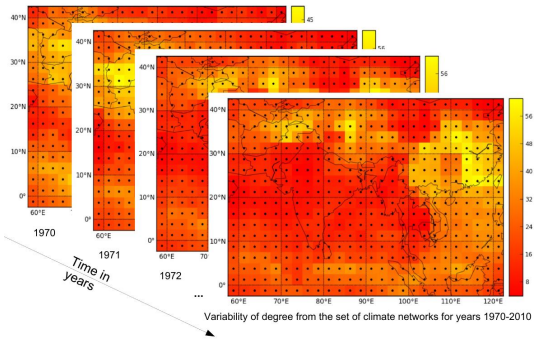

Figure 15: Degree fields for 40 yearly climate networks constructed for temperature time-series for (1970-2010). Illustration for *degree variability* measure between various degree fields. All climate networks are constructed for the same link density of 5%.

which have been discovered during the last years using novel methods of physics and mathematics [48, 137, 184]. The motivation to study the climate variability comes from the recent studies which have shown the supporting evidence of climate change [267]. The question of variability and predictability of climate and Earth system dynamics has always been a challenging topic. Given the complexity of the inter-relations between the subsystems that constitute our climate, it is important to approach the problem from an interdisciplinary perspective, e.g. using climate networks [81, 100, 174, 261]. The climate networks have been used for detecting long range correlations, or teleconnections [19] and studying phenomena such as the El-Nino/Southern Oscillation (ENSO) [57, 261], and the Indian Monsoon system [162, 212]. In particular, Tsonis and Swanson found changes in the global network topology during El Nino events with significantly fewer links and lower clustering coefficients and inferred a lower predictability for El Nino over La Nina years. Using climate networks [100, 269] also found ENSO influence on regional atmospheric processes in non-ENSO regions. Since data is available [193], we construct the network directly from the available climate data time series. Understanding and modeling climate variability leads to identify the main "ruling elements" of this complex system. For this reason Asian monsoon (or more specifically Indian monsoon) and ENSO are one of the main elements of climate system.

Asian Monsoon Data. Ensemble of correlation networks are generated for the daily NCEP/NCAR reanalysis temperature anomaly data [193] for the Asian monsoon domain for the years 1970-2010 C.E. The spatial resolution was $2.5° \times 2.5°$, which covers area between $2.5°$S to $42.5°$N and $57.5°$E to $122.5°$E, resulting in time series for 468 nodes. Networks were constructed using Pearson correlation in windows for each full year and by thresholding the correlation matrix such that we obtain a link

density of 5%. As the result, 40 climate networks were obtained and analysed using novel evolving networks measures, Fig. 15.

3 Methods to characterize evolving networks

It is clear, that methods and algorithms to detect changes and differentiate graphs in network ensembles are needed. Some types of evolving networks models are illustrated in [68, 122]. Recently a number of artificial networks of such a kind came into existence, which revealed the questions about topological properties, complex processes occurring in the evolving networks. I list some possible methods to characterize the networks evolution for the network ensemble denoted as $\{G_i\}$ in Table 2.

This list can be continued by all possible combinations of this methods. I introduce new methods, i.e. the 5^{th} and 6^{th} methods in Table 2 and compare them with some of the existed approaches.

3.1 Quantitative indices to estimate networks evolution

The Common Component Evolution Function

The 5^{th} method in the Table 2 gives quantitative characteristics for the network evolution using basic set theoretical notions. Note, that I consider unweighted and undirected networks without self-loops. n nodes have fixed numeration as it is shown in Fig.16. The linking structure is given in the adjacency matrix A, a binary $n \times n$ matrix with zeros on the diagonal. An element is non-zero, $A_{ij} = 1$, if and only if the vertices i and j are connected, and zero otherwise.
Let us consider a linearly ordered set of T evolving in time networks: $G_1, ..., G_T$.

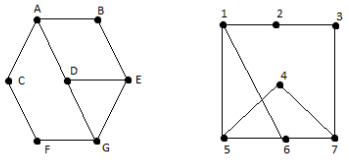

Figure 16: Two networks can be isomorphic having different topological realisations on the plane. It is important to mention that the definition of the *common component* of networks, embedded in the plane, depends on the numeration chosen for nodes.

Definition. Then the *common component network* , $\mathrm{CC}(G_i, G_j)$, for two of these networks G_i and G_j is a network on the same nodes, where the set of edges is present in both original networks.
If G_i and G_j have adjacency matrices A_i and A_j, the number of edges in the common component network $\mathrm{CC}(G_i, G_j)$ is the number of non-zero elements above the diagonal

Table 2: Methods to characterize the networks evolution

#	*Method to characterize the networks evolution*
1.	Compare the *degree distributions and link length distribution* for each network $\{G_i\}$, one of the possible methods of quantification of the degree distributions in complex networks is demonstrated in [3];
2.	Calculate *degree-degree correlations* between the pairs of nodes $deg_k(G_i)$ where the degree value $deg_k(G_i)$ for node k is calculated for G_i for one of $i = 1...n$ networks. It has been highlighted in [31, 65], the degree-degree correlations and degree distributions reciprocity are important characteristics in the network ensemble;
3.	Estimate the changes in the frequency of the networks *patterns or motifs* [168];
4.	Perform *the eigenvalue analysis* for each of the $\{G_i\}$ networks. Eigenvalues of the network adjacency matrices are directly connected with identification of the *network giant components*. The main idea of this approach is that if the giant component of network G_i and G_{i+1} is changing, the changes are significant;
5.	Use *the common component method (common component evolution function)*, defined in Subsection 3.1, generalized Hamming distance between network adjacency matrices and other set theory based network measures [209, 253];
6.	Collate *the degree and other network fields using qualitative comparison* of the degree and clustering coefficient fields, defined in Subsection 3.2. This can be calculated by substraction of the degree fields [253].

in the binary sum of adjacency matrices A_i and A_j. This common component network can be generalized for any $k+1$ networks by induction:

$$\mathrm{CC}(G_i, ..., G_{i+k}) = CC(CC(G_i, ...G_{i+k-1}), G_{i+k}) \ .$$

Now the *common component function*, $\mathrm{CCF}(G_i, \ldots, G_{i+k})$, counts the number of links in a common component network of k networks:

$$\mathrm{CCF}(G_i, \ldots, G_{i+k}) = ||CC(CC(G_i, \ldots, G_{i+k-1}), G_{i+k})|| \ ,$$

by $||G_i||$ I mean the number of links in the network G_i, and the common component function $\mathrm{CCF}(G_i, G_k)$ gives the number of coinciding edges in the graphs of G_i and G_k, $i, k = 1, 2, \ldots, T$. I set $\mathrm{CCF}(G_i) = \mathrm{CCF}(G_i, G_i)$. The common component function $\mathrm{CCF}(G_i)$ takes values in $[0, max_i CCF(G_i)]$ and is in the following normalized to $[0, 1]$ using the maximal number of links in the networks.
Idea of the common component method:
I take the mean over the CCFs with the same time lags to estimate the non-normalized *common component evolution function*, CCEF^*, as

$$CCEF^*(\delta) = \frac{1}{T-\delta} \sum_{\mathrm{i}=1}^{T-\delta} CCF(G_i, G_{i+\delta}) \ , \tag{6}$$

where δ is the time lag between the networks, and $\delta \in [0, T-1]$. Therefore method of CCEF is inspired by the covariance estimation [46] for networks evolution. The maximum value of the CCEF^* is given by $\mathrm{CCEF}^*(0)$ for zero lag, as an average number of links in the set of network

$$CCEF^*(0) = \frac{1}{T} \sum_{\mathrm{i}=1}^{T} CCF(G_i) \ , \tag{7}$$

and I use it to obtain the normalized *common component evolution function*

$$CCEF(\delta) = \frac{CCEF^*(\delta)}{CCEF(0)} \tag{8}$$

which I will use exclusively in the following. As an estimation of the CCEF uncertainty I use the standard deviation over all CCEF values.

3.2 Qualitative measures to estimate networks evolution

The 6^{th} novel method in the Table 2 describes the network evolution from the qualitative side. While some measures are perhaps best applied to networks aggregated over chosen time periods, e.g. the time-dependent degree of a node can be computed as the number of links activated within some time window. Other properties are directly influenced by the order of link activations, defined in [122].
Let us define the *network measure for the network G embedded into space X*. For

simplicity let us assume that the nodes are regularly distributed on the planar grid. For each link e_{ij} from node i let us define a weight proportional to the distance in space X. Then summing up all the "weighted-geographically" degrees of the node, we observe not just the number of connections of the nodes, but also the relevance of the connections the node has from the transport prospectives.
Despite the differing shapes, these two graphs are isomorphic. The result from solving this could also reverberate beyond computer science, such as allowing chemists to determine whether complex molecules have the same bonding structure, Fig. 16. The problem of solving the graph isomorphism problem, is one of the most costly problem. The recent solution algorithm was proposed by L.Babai.
Approach for detecting significant patterns in networks is related to motifs. A network motif is an equivalence class of subgraphs that is overrepresented in terms of its cardinality with respect to some null model in a network, i.e. a larger number of such subgraphs can be found than in a randomized reference system [122]. Usually, the configuration model that conserves the degree sequence but otherwise randomizes the network is taken as the reference system. The over- or underrepresentation of certain subgraphs can be related to the function of the system, especially in directed networks where the subgraphs forming motifs can be associated with e.g. information processing tasks. There are several ways to extend this concept to temporal networks. The most straightforward approach is to look at snapshots of the network taken at different points in time, or alternatively aggregated edges over a period of time, and count the different subgraphs in these snapshots. Suggested constraint of *common component function* can be extended to analyze evolving networks of different types. Networks, embedded and non-embedded in space, are treated in the same way with a difference that networks, embedded in space, have additional node attributes.
Finally, I define measures, generalized from static to evolving networks.
Definition. *Degree variability (DV)* is a network measure on an ordered set of k networks $\{G_i\}, i \in (1,k)$ with fixed set of s nodes. Let us define deg_j^i as degree of $j-$th node in i-th network. Then the *degree variability (DV_j) of node* $j \in (1,s)$ is the average deviation of deg_j^i:

$$DV_j = D(deg_j^1, ...deg_j^k) = \frac{\sum_{l=1}^{k} |deg_j^l - \overline{deg_j}|}{k}, j \in (1,s), i \in (1,k). \quad (9)$$

Definition. *Evolving average path length* (EAPL) is function defined on ordered set of k networks depending on the network index $i \in (1,k)$: $EAPL(i) = APL(G_i)$, where $APL(G_i)$ is average path length for the G_i network.
Definition. *Evolving transitivity (ET)* measure on ordered set of k networks is defined in analogy to EAPL.

4 Applications of temporal network measures to evolving networks

Another roof, another proof.

Erdos Renyi

First I address the question of *qualitative characterization of evolving networks* which allows to capture more features of the system, than just one network measure. In the next Subsection 4.1 I *describe the evolving networks quantitatively*, in order to find a mathematical function characterizing temporal climate networks. The application of the time-evolving indices characterizes the variability of the climate networks. In [209] the temporal yearly global climate networks have been analysed. I suggest additional temporal climate networks indices and study the regional climate networks demonstrating its applications on random and networks from the temperature data over the India subcontinent. Further the term *evolving (in time) networks* is used as synonym to *temporal network* if not explicitly stated otherwise.

4.1 Index for random evolving networks

While speaking about random networks we should keep in mind that a particular network we observe is only one member of a statistical ensemble of all possible realizations. Hence when we speak about random networks, we actually mean statistical ensembles [67]. Therefore the results presented for Erdos-Renyi random graphs are made for the ensemble, i.e. several network realizations. This result can be generalized for any random networks ensemble including networks with scale-free property.

To test method introduced in Subsection 3.1 I generate a set of T Erdős-Renyi graphs [77] with fixed n nodes and a fixed connection probability p. I artificially impose a linear ordering on the set, such that it is possible to index them with $i \in (1, T)$. I compute the CCEF for Erdős-Renyi-graphs with 100 nodes and link probabilities of 0.3, 0.5 and 0.9. The resulting functions, shown in Fig. 17, decrease from 1 to a plateau at $\mathrm{CCEF}(\delta) \approx p$ for $\delta > 0$ for each link probability p.
For this example it is possible to compute analytically the expected CCEF, since for $\delta = 0$ each network is compared with itself and therefore CCEF $= 1$.
Proposition II. Analytical formula for CCEF for ER-networks ensembles:
$f(p) = \frac{p^2 n(n-1)}{pn(n-1)} = p$.
Proof: For all other values of δ two random matrices with n nodes and connection probability p are compared. Then the number of totally possible links is $n(n-1)/2$, and the expectation value of the number of links in each of the networks is $pn(n-1)/2$. As the probability of each of the edges in one network to also appear in the other network is p, the total number of common links is $p^2 n(n-1)/2$, which with the

normalization leads to $f(p)$, the ratio of total number of common links and the expectation value of the number of links to be

$$f(p) = \frac{p^2 n(n-1)}{pn(n-1)} = p \ . \tag{10}$$

The CCEF for each linking probability therefore lies close to the expected value p.

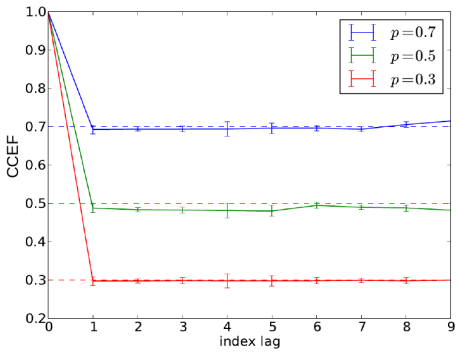

Figure 17: Estimation of variability of random networks using common component evolution function (CCEF). CCEF for indexed set of random Erdős-Renyi networks with 100 nodes and different linking probabilities p marked by different colors. The dashed lines correspond to the analytical CCEF levels, the errorbars give the 1σ standard deviation for each index lag.

4.2 Network measures for evolving networks embedded in space

> What, Hex asks, is a dimension? So Arthur explains: A point is zero-dimensional; a point moving straight traces out a one-dimensional line...
>
> "Flatland", Edwin Abbott Abbott.

Many, if not most, complex networks of interest are spatially embedded. Consider, for example, social networks, infrastructure networks such as the internet, road, and other transportation networks or functional networks in neuroscience and climatology [194]. In [63] the measures and models specifically designed for spatially embedded networks (even more generally speaking spatial networks) are described.
The network embedded in some space is closely related to the field of mathematical topological graph theory, first discovered in 1736 by Euler with his famous ($V-E+F=2$) and then was dormant for 191 years. The number of problems on the topological characteristics of the networks have accumulated.
Let us first consider one fixed network and define set of networks $\{G_i\}$ embedded into space S. One of the most straightforward approaches to describe variability of

networks (non)-embedded in space is to characterize how far each network G_i deviates from the common component network (CCN), and what are the main patterns which are formed by these deviations of each G_i, i.e. to use qualitative measures. Close approach to the previous one is to find quantitative characteristics, indices, such as, for example, CCN, fluctuations of the global transitivity coefficients calculated for each network for different time-periods.
In the previous section network measures were applied to Erdos-Renyi network non-embedded in space. Locality properties add some new information to the network, therefore one needs to characterize network using additional functions. Recall that $\{G_i = (V_i, E_i), i = 1, ..., n\}$ is a numerated set of networks, where V_i and E_i are corresponding set of nodes and edges of the network G_i.
Further I apply the CCEF measure to three different types of networks *embedded in space*, test models: flow-networks, and correlation networks from START model and from the climate data for the Indian subcontinent, Subsection 2.4. In order to complete the analysis I apply the DV, EAPL, ET measures to climate networks in Subsection 3.2.

4.3 Estimation of global variability for evolving networks

The results of applications of network measures for the different ensembles of networks are followed by the comparison of the CCE function for different network evolution.

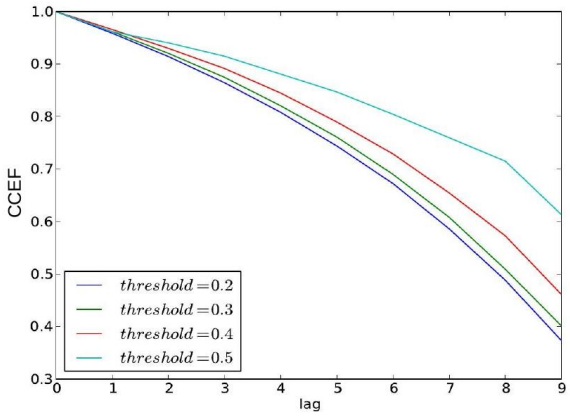

Figure 18: Estimation of variability of parametrized flow-networks using common component function (CCEF). CCEF for the indexed flow-networks with increasing flow-width parameter c, and for different threshold values of adjacency matrices of the flow-networks.

CCEF for flow-networks I computed the CCEF for flow-networks with linearly increasing flow-width parameter c. As Fig. 18 shows, the common component size decreases monotonously with the width parameter difference of the networks. The

higher the threshold of the correlation matrix is, the faster the CCEF decays but the general shape does not change.
Unlike random networks, the flow network CCEF level is, within limits, not related to the threshold value but displays a deterministic decrease of network similarity: two flow-networks separated by bigger index lag have less links in the intersection, hence the common component function $\mathrm{CCF}(N_i, N_{i+\delta})$ decreases with the growth of δ, and the intersection of two flow-networks decreases with the difference in the width parameter. Furthermore I find that the links in a network set with higher threshold are more persistent.

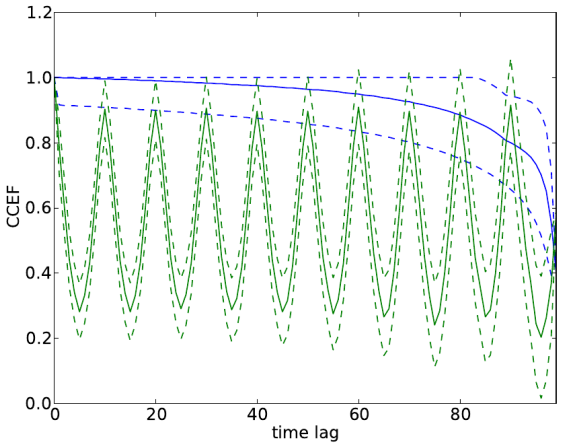

Figure 19: Estimation of variability of correlation networks from the START model using common component evolution function (CCEF). CCEF for networks from the START model with periodic (green) and linearly increasing (blue) forcing parameter F. The error bars indicate the standard deviation of the CC size estimates.

CCEF for START model correlation networks The START model undergoes a more distinct transition from a network with two distinct parts through a connected stage with three regions to one single component [213], in response to a single forcing parameter F. To characterize the CCEF response to different network evolution patterns I use two test cases, in which I vary F from its minimum to its maximum. In the first example, the forcing parameter is varied linearly along time. The CCEF response is a slow decline from its maximum $\mathrm{CCEF}(0) = 1$ to a minimum value $\mathrm{CCEF}(99) \approx 0.4$, as shown in Fig. 19. In the second test the forcing parameter was varied periodically as a function of time, $\vec{F} = \sin(\frac{2\pi}{P}\vec{t})$, with $P = 10$. In response to the sinusoidal forcing, periodic behavior is also observable in the CCEF and with the same period as the forcing parameter.
The START model examples illustrate the distinct difference between slow, linear changes of the processes generating the networks over time - and periodic, rapid

transitions. While in the first case the CCEF decreases slowly, and only considerably for large time difference, in case of periodic and rapid transitions the CCEF response is also periodic over the time lag. In this case it is particularly important that the time window a single network corresponds to is sufficiently small compared to the ongoing evolution to avoid aliasing effects which would occur in case of window width as a multiple of the forcing period and to be able to detect the changes at all.

Estimation of Monsoon Variability using evolving networks

I used the CCEF to investigate the evolution of climate networks from observations. The networks were constructed using a link density of an annual basis for 40 years, 1970-2010 C.E. The obtained CCEF in Fig. 20 is reminiscent of the Erdos-Renyi networks with an initial quick decline followed by a plateau.

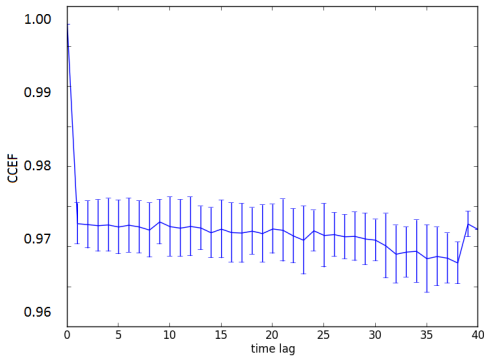

Figure 20: Estimation of variability of climate networks using common component function (CCF). CCEF of annual climate networks for the time period 1970-2011 C.E., error bars are presented as CCEF standard deviation of the respective time lag in years.

However, while in the ER-case, Fig. 17, the baseline is equal to the set link density, it is significantly higher than the link density for the climate networks. Thus I conclude that a high degree of persistence and a low amount of spatio-temporal variance can be found in climate networks from the Asian Monsoon domain at annual time-scale.

Evolving network measures can be considered as generalized static network measures: degree, betweenness, or global measures such as average path length and transitivity [21]. For analysis of the annual variability of the climate network of the Asian monsoon domain, I use measures for estimating the persistence of evolving in time networks, described in [253] and [209]. In particular, I use temporal network measures introduced in Section 3: *Evolving average path length (EAPL)* and *Evolving transitivity (ET)*, based on static network measures [191], and calculate them for climate networks of the Asian monsoon domain defined in Subsection 2.4.

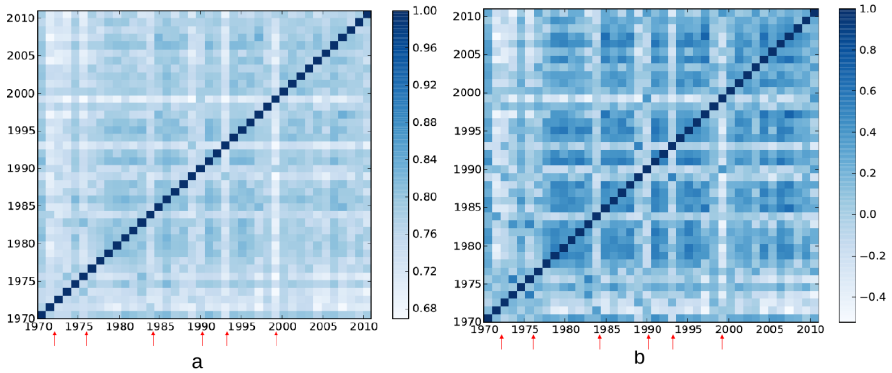

Figure 21: Common-links-recurrence-diagram (a) and correlation matrix of the common link evolution (b). Each point (i, j) in the diagram (b) corresponds to the value of the correlation coefficient $corr(i, j)$ between the common component functions $CCF(G_i, G_k)$ and $CCF(G_j, G_k)$. Lines with low values (marked at the bottom with arrows) are observable around strong ENSO years.

Network measures to describe annual variability of climate networks. Annual variability of the Indian monsoon has large affect on the agriculture and economy of the region. Hence I am interested in tracking temporal changes in the spatial structures of the climate networks of the Asian monsoon on annual time scale. Analysis of the annual variability of the evolving climate network of the Asian monsoon region allows us to conclude that a highly non-random, deterministic general structure is present in the network on which the inter-annual variability is imprinted [179, 253]. The annual climate networks variability can be explained by dominant influence of the topography of the region on the climate network as well as regular monsoon effect, or by big climatic event such as El Nino or La Nina [24, 100, 252]. In order to investigate this question I calculate *evolving average path length* and *transitivity* Fig. 22 for evolving over time climate networks constructed for Indian Monsoon region for the period (1970-2010) Fig. 15. Most of the peaks of the EAPL correspond to big El Nino (EN) years, while troughs of ET correspond to La Nina years according to classification of EN in [145]. This fits well with the results of annual variability for global climate networks [209]. The Seasonal variability also can be described using measures introduced in Section 3 when the climate networks are constructed separately for each of the season period [63].

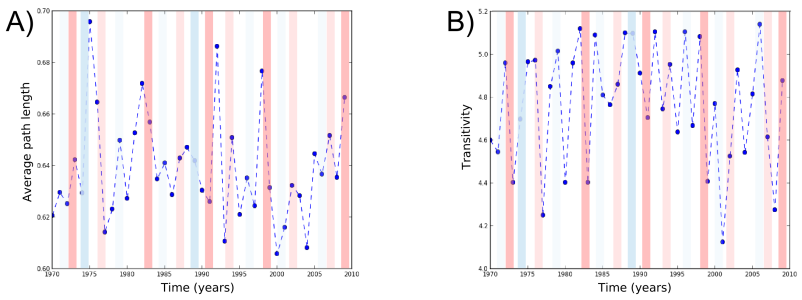

Figure 22: Evolving average path length (A) and transitivity (B) network measures are calculated for each neighboring pair of yearly climate networks constructed for the period (1970-2010). Most of the peaks of EAPL correspond to big El Nino (EN) years (red vertical bars), troughs of ET correspond to La Nina events (blue vertical bars) where color intensity of the bar corresponds to EN event strength. Data has spatial resolution $2.5° \times 2.5°$ covering the area between 2.5°S to 42.5°N and 57.5°E to 122.5°, i.e. 468 nodes.

5 Discussions and conclusions

Ship with Leviathan for forty years Until an
isle in Space looms up to match your dreams
unpublished poem of Ray Bradbury

I presented novel methods to characterize evolution of temporal networks and applied it for various types of network ensembles, described in [63, 179, 253]. With model tests in Section 4 I established that it is possible to use presented method to distinguish random, deterministic and periodic evolution behaviors in a set of networks. The new quantity to measure variabilty and persistence in networks is suitable for different types of network ensembles where the network set may be linearly ordered by time - or by parameter difference. The *main conclusions* of Chapter II are the following:

- 1. *The CCEF enables to investigate the evolution of linearly ordered, or evolving, network sets quantitatively.* I tested its response to three different types of model networks and find, that their response enables us to characterize their spatial and temporal evolution from different sides.

- 2. *The year-long daily temperature anomaly networks of the Asian Monsoon domain show a high degree of spatio-temporal persistence.* While the general shape of the CCEF agrees with that for the Erdos-Renyi random networks, the CCEF oscillates around a much higher level. In order to investigate, whether the main changes in climate networks happened due to changes in degrees of "supernodes" (nodes with higher degree), I estimated the degree variability (DV) measure, shown in Fig. 23(a), for networks ensemble from 40 yearly correlation networks, Fig. 15. In order to investigate, whether variability of nodes' degrees are correlated with each other I constructed degree correlation measure, shown in Fig. 23(b), for the same 40 yearly correlation networks. These results point towards a highly non-random, deterministic general structure in the network on which the inter-annual variability is imprinted. Links in this network are comparatively stable but loose some of their stability when the external disturbance of an El-Nino-event is added. This agrees well with the findings of [100, 252], who showed that, for global networks, fluctuations, or "blinking" of links could be related to the global signature of ENSO variability. Improving our understanding of the Earth's complex climate elements, such as Indian Monsoon, has a huge economic and social impact for present and future generations, and can underpin advances in areas as diverse as energy, environment, agricultural and marine sciences.

- 3. *Common component framework for functional networks evolving in time is a promising and useful tool for analysing spatial and temporal transitions in various climatic phenomena* [100, 212]. In particular, evolving climate networks have been used to study seasonal and annual variability of the Indian Monsoon system one of the global climatic phenomenon affecting life and prosperity of 1/4th of the world's population [30, 161, 253]. On seasonal time scale, it is crucial to identify spatial structures of synchronicity of extreme rainfall events over the Indian monsoon domain, as extreme rainfall events are the main causes of the devastating floods on the subcontinent. On annual time scale, variability of the Surface Air Temperature (SAT) is of a great interest, as it influences total amount of rainfall and it's spatial distribution during the monsoon season.

  Understanding the variability and evolution of the Indian monsoon and its interaction with ENSO remains one of the most vital questions in climatology. The interactions of these climatic components were studied from the climate networks perspective, reveal influence of Western Disturbances and Westerlies on the synchronicity of the extreme rainfall events over the Indian subcontinent, seasonal and annual evolution of the spatial structures and dynamics of extreme rainfall and temperature climate networks over the Indian monsoon domain, and the influence of ENSO on the monsoon system.

- 4. *Introduced method of evolving networks characterisation is based on the common component function* for networks embedded and non-embedded in space. *This measure determines a constraint generalizable to the broader list*

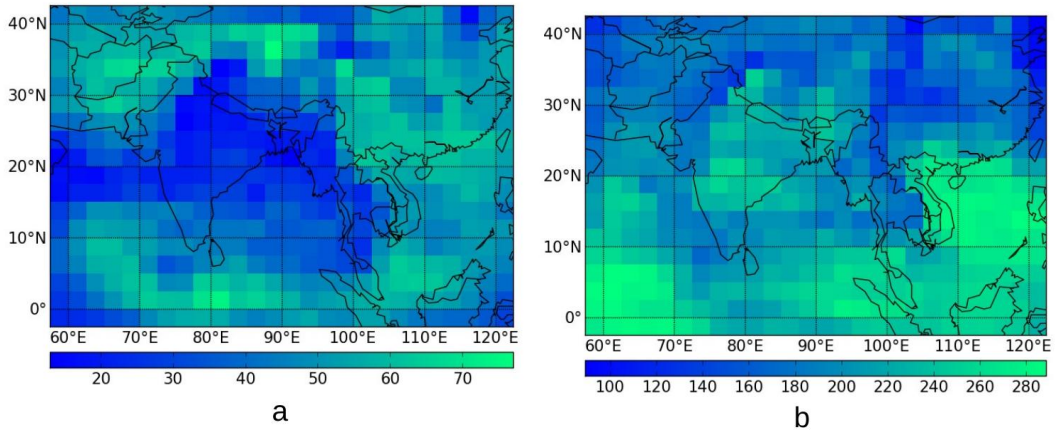

Figure 23: The variability of degree fields for climate networks, constructed for each year in the period (1970-2010) (a). The degree measure for the network constructed for correlations between degree variability in each node (b). Climate networks are constructed for the period (1970-2010).

of network types, for instance, for weighted and directed graphs. The method can be extended in a straightforward manner, and it currently requires that the node structure and link density remain constant.

Outlook: analysis of time-evolving networks

In this chapter the evolving network measures were inverstigated from different perspectives. Temporal networks continued to be in the center of interest till nowadays [219], for instance, the algorithm using greedy random walks. Theory of temporal network evolution was recently applied to molecular dynamics and detection of connected regions in the ocean [86], and World-Wide-Web analysis [67], where also statistical ensembles of random networks were analysed. Many different random network concepts have been developed after seminal work by [77], where networks generated using some fixed mathematical rules exhibit phenomenon present in many physical systems [1]. There are still many unanswered questions on evolving networks, and in particular evolving functional networks:
1. What are the criteria to find *the best combination of methods to characterize the evolution of networks*?
2. *How to validate the mechanism of "synchronisation" in climate networks* shown in Fig. 21? Synchronisation is a very important phenomenon observed in many natural systems [29, 128, 204]. Some mechanisms of the persistence in climate networks and synchronous break down of links in climate networks were discussed in [24]. Temporal and spatial variability of climate, and thus climate network structure, are of increasing interest considering ongoing environmental changes. Climate networks as evolving in time are still an open subject. The spatial-temporal developments in a given network set can be too complex to be captured by eye, and systematic approaches to quantify

changes are needed. While in [24] the origins of the climate network stability were investigated, such as the spatial embedding and physical coupling between climate in different locations using the correlation between correlation matrices, other studies describe how the network graph is changing over time to understand the behavior of the underlying dynamical system.

3. *How various types of functional networks are connected between each other?* Such connection can be performed by method of diffusion maps [51], which bridges the correlation and recurrence networks for finding meaningful geometric descriptions of data sets. If correlation networks are constructed in the space of time-series, i.e. T-dimensional space, where T is length of time series, recurrence or diffusion maps are

4. Furthermore, *suggested temporal network measures can be applied to temporal functional networks* such as random network models [21], analytically derived flow-networks [177, 229, 253] and paleoclimate networks [212]. Moreover, the common component analysis can be generalized to nodes with labels in order to tackle more general class of networks.

4. *Topological properties of evolving networks (or spectral properties of the adjacency matrices) can be considered in combination with introduced method.* The spectral properties of the random networks have been studied [68], while for the ensembles of correlation matrices, Subsection 4.2, detailed analysis is missing. Topological properties of the networks are further discussed in Chapter IV.

Theoretical foundation of correlation networks

6 Introduction to correlation networks

6.1 Correlation networks from data and dynamical systems

> And he showed towards the upstream of the river, there were the hills...
>
> Anrie Bosco "Small boy and the river"

In this chapter I introduce a method of correlation network construction for dynamical systems and derive theoretical insights into network characteristics relating them to properties of dynamical systems. The main subject of Chapter II was to develop the method to characterize evolving networks and apply this method to correlation networks analysis. In this chapter I construct correlation networks for systems governed by advection-diffusion dynamics, approaching the question of interpretation of correlation network measures from a general perspective.
The main focus of this chapter is the novel discrete *flow-networks method*, which I developed as a new technique for dynamical systems analysis and understanding features of correlation networks [254]. This approach suggests a novel view on *advection-diffusion dynamics* with time-independent [147] and time-dependent [254] underlying systems. Applications of the flow-networks method are demonstrated for non-autonomous systems with noise, external heating and temperature decay. The advantage of such method is that it is applicable for different types of dynamical systems.
First, in Subsection 6.2 I give a brief *historical overview on network methods* in connection to several branches of physics. In Subsection 7.1 I present novel discrete *flow-networks method*, the *method of a correlation network construction* for given flows, discretising the advection-diffusion equation (ADE) on a regular grid. In Section 8 the method of flow-networks is demonstrated for systems determined by time-dependent velocity fields with various mixing properties and dissipation rates. I apply the method of flow-networks analysis to a system, which exhibits regimes with large-scale mixing for certain values of control parameters. In the Subsection 8.4 I present techniques allowing to characterize network measures, constructed from a physical system. Moreover, I consider a geometric properties of networks, which are embedded into a metric space with corresponding spatial alignment and distribution. It is important to mention that a term "flow-networks method" is

used for the method of direct construction of correlation networks from flows, which has to be distinguished from the so-called "Lagrangian flow-networks method" [215]. Additionally in this chapter, I compare the flow-network method with other techniques to study transitions in dynamical systems such as "continuous flow-networks" [177] and "Lagrangian transport flow-networks" [228]. In addition, it is important to mention that "flow-networks" is a general term which should not be compounded with "flow networks" defined by transportation of flows on networks [33].

General flow-networks method, introduced here, allows to broaden applications in order to describe a broader spectrum of systems described by linear differential equations.

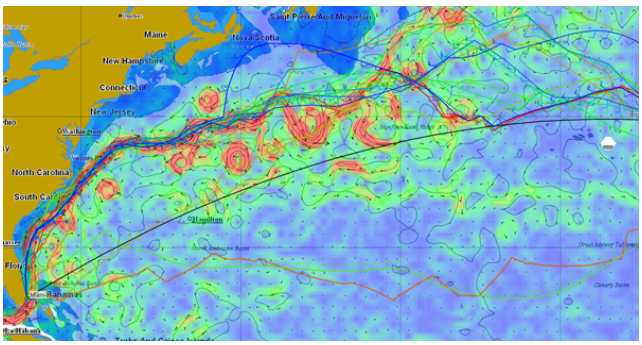

Figure 24: The Gulf stream current in the Atlantic ocean has a complex turbulent structure. The meandering flow is periodically changing in time [38]. The image source is www.tidetech.org.

6.2 Historical overview over network theory aspects for analysis of dynamical systems

A story that never ended!
"Never ending story" Michael Ende

A brief summary of network theory concepts and their applications have been presented in Chapters I and II. Complex networks have been successfully applied to various systems [44, 157]. In parallel to the development of graph theory (period from the beginning of 1730s) the interest to of networks as a tool to analyze dynamical systems was constantly progressing [242]. Applications of network structures explain collective behavior and critical phenomena in different systems, such as synchronisation of fireflies [236], or the phenomenon of "six degrees of separation" in society and a small-world property of neural networks.
The idea of connecting network theory with thermodynamics was presented more than 40 years ago, when the term "network thermodynamics" was first introduced [197]. There techniques of network theory were generalized in order to include into

consideration irreversible thermodynamic systems. Nevertheless, theory developed in [197, 202] was merely a graphical representation of conservation equations not focusing on connection between complex networks structures and dynamical systems properties. Another concept which takes central place in a large number of discriplines is the concept of *flows*, investigated from various perspectives including graph and knot theory [95, 185]. In [82] the thermodynamic properties of networks based on their linear and non-linear characteristics were studied, which brought new insights about the non-equilibrium steady states for various boundary fluxes in thermodynamic data. The pioneering fundamental research in this direction was made by Prigogine in 1970s [207], who studied non-linear non-equilibrium self-organizing systems with dissipation. In his work the theory on dissipative structures was developed together with the "missing bridge" between thermodynamics and general dynamical systems theory. Analytical results on the theoretical background for the connection between the topology of the attractor and the dynamics of the system were studied in KAM-theory [235].
At the same time during the period starting from 1960 till nowadays attention is only increasing to problems of dynamical systems theory [9], some of which were translated to the language of complex networks and stochastic networks were developed [28, 42]. Structure of complex networks serve as mathematically tractable model of complex systems. For instance, one of the main milestones of ergodic theory [7], which has been recently intensively studied from the perspectives of complex networks [230, 274], is the problem of detection of transitions to chaotic regimes in systems with low number of degrees of freedom. Another example of emergence of network theory with other fields of physics is analysis of ordering dynamics on network models [119, 240], more details on this are placed in Chapter IV. Theories of disordered dynamical systems, stochastic theory and network theory have been merged together to answer common questions [135, 154]. In particular, transport properties of system dynamics [76] has been extensively studied using so-called transfer operator approach [85]. In brief, an approximation of a transfer operator P can be built using transition matrices. Any stochastic matrix P (sum of matrix elements for each column and row equals one) denotes a transition matrix P, where each matrix entry p_{ij} defines a transition probability of a switch between i and j states of a Markov chain [84]. Applications of transfer operators analysis to climatology have been recently presented in [48, 246]. One should emphasize that the transfer operators approach and the Lagrangian flow-network networks [215, 225, 229] both are closely related: a transition matrix for Lagrangian flow-networks is estimated by so-called Ulam's method [88, 228]. According to Ulam's method, transition matrix P for a given set of particles trajectories in a given velocity field is a discrete approximation to a transfer operator, or Perron-Frobenius operator. As the result, flow-network measures characterize marine connectivity and transport between different regions of a fluid domain [228], or transport barriers [79]. The network approach allows to link dynamical properties of a complex system with a network topology.
Functional *flow-networks* [87, 228, 254] and *correlation networks* [62, 100, 173] have been relatively recently developed as a new instrument for data analysis. These func-

tional networks also brought up new challenging questions. One of such questions is, for instance, how to interpret measures of correlation networks, which are constructed from observables such as temperature, velocity or pressure, (this has been also discussed in Chapter II). Links in a correlation network are defined between two locations, if correlation between time series for one fixed observable is higher than a threshold. Therefore, network links are considered to imply information or heat exchange [81, 216]. However, a relationship between oceanic or atmospheric flows and structure of correlation networks still remain unclear. In [147, 177, 254] theoretical approaches have been introduced to verify the relation between underlying flows and correlation networks from temperature variation. While some approaches examined stationary flows [147, 177], in [254] the theoretical flow-networks method was generalized to *time-dependent flows*, which exhibit more complex and rich dynamics than stationary flows. Importantly, time-dependent flow dynamics is present in all ocean currents, such as the Gulf Stream Current with its turbulent meandering structure, Fig. 24. Hence, *flow-networks method for time-dependent underlying systems* significantly expands applications of correlation networks analysis. As the result, application of time-dependent flow-networks method admits deeper understanding of the physical meaning of correlation network properties.

6.3 Research questions: construction and analysis of correlation networks

> Tell a man there are 978,301,246,569,987 stars in the sky, and he will believe you. Show him a "Wet Paint" sign, and he will check and get his finger stained
>
> Bernard Show

The definitions for several types of *functional correlation networks* are given in Chapter II. However, as has been mentioned before, analytical approach to correlation networks is missing. In order to give correct interpretation to correlation network measures I analyze the correlation matrices from dynamical systems exhibiting complex predefined behavior, using the language of complex networks. This idea is schematically illustrated in Fig. 37.

Not all general properties of correlation networks are fully understood. Here I consider a system with advection-diffusion dynamics with additional degree of freedom, introduced by time-dependency of advective term. This opens up a new challenging problem: to develop methods of characterizing time-dependent underlying system, transitions in flow dynamics. But wait, why exactly advection-diffusion processes? Advection and diffusion dynamics is present in driving sources of many processes on our Planet. Altogether reaction, advection, diffusion were widely studied using different techniques [155, 187], and successfully applied in different areas. For instance, it has been identified that the existence of a smooth-filamental transition in the concentration patterns depends on the relative strength of the stirring by the

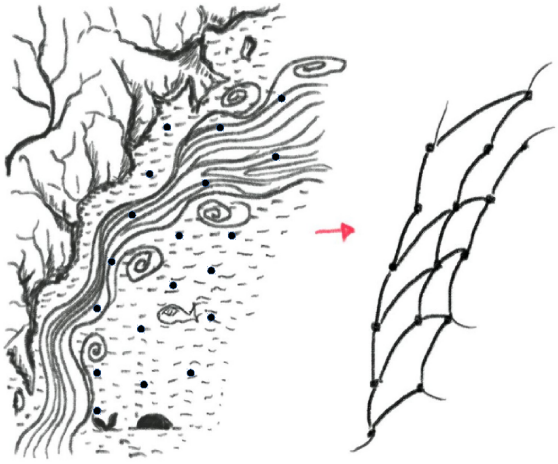

Figure 25: The interpretation of flow-networks concept: nodes are parts of croase-grained continuous system, links imply information or heat exchange between nodes. Then network topological features identify characteristics of the flow systems, for instance, mixing in the flows.

chaotic flow and the relaxation properties of planktonic dynamical system. I refer to [85, 172, 229] for overview of traditional methods of analysis of such systems and below introduce idea of novel flow-networks techniques.
Recently a large variety of models have been investigated using flow-networks paradigm [178, 189, 228, 254, 257], where problems from dynamical systems, in particular, identification of critical transitions and shifts between dynamical regimes, were formulated in terms of complex networks theory. One may consider a flow system from different perspectives, which is explained below.

Flow systems analysis from Eulerian and Lagrangian perspectives

The hydrodynamic equations of motion, such as advection and diffusion, can be studied from Eulerian and Lagrangian perspectives. In *Eulerian approach* one deals with velocity, pressure and density fields in the space domain covered by the fluid, while in *Lagrangian approach* the trajectory of each fluid particle is considered [52]. Let us denote the Eulerian velocity field by $v(x,t)$, and the motion of a fluid particle $x(t)$ with the initial condition $x(0)$ is determined by the differential equation

$$\frac{dx}{dt} = v(x,t) \tag{11}$$

where $v(x,t)$ is the Eulerian vector field defining the dynamical system. If the solution of Eq. (11) has a sensitive dependence on initial conditions, and initially nearby trajectories diverge exponentially fast, one speaks of Lagrangian chaos. Eulerian

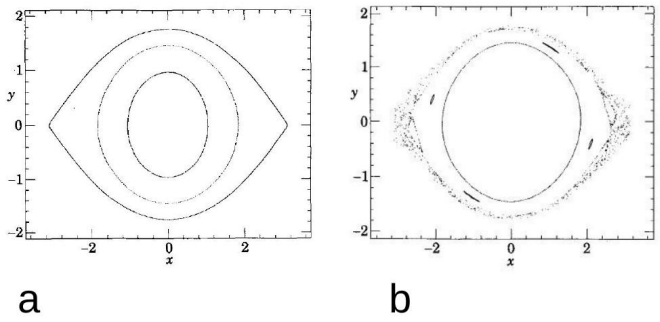

Figure 26: Orbits of Kuznetsova flow, defined by Eq. (12) from [52], for parameter values $h = 1, \mu = 0, \omega = 1$ (a), $h = 1, \mu = 0.01, \omega = 1$ (b).

and Lagrangian points of view are in principle equivalent, however, the simple formal relation between these approaches doesn't provide the exact information about Eulerian system starting from Lagrangian one and the other way round.
For instance, let us consider the explicit time-dependent streamfunction $\Psi(x, y, t)$ of a system, which exhibits chaotic motion, Fig. 26. The streamfunction sets the velocity equations:

$$\frac{\partial \Psi}{\partial y} = -h\frac{sinh(y)}{(cosh(y) - cos(x))}; \quad \frac{\partial \Psi}{\partial x} = h\frac{sin(x)}{(cosh(y) - cos(x))} + 2\mu y cos(2\omega t)\omega h \qquad (12)$$

where h is the dimensionless time unit and $\omega = \frac{l^2}{\pi \Gamma h}$ for l indicating the period of vortex series in x-direction [9, 52]. It's known that making small perturbations to the system, as in this example, can introduce transitions to chaotic regimes. The approaches for studying the influence of small perturbations on solutions of the system are, for instance, the response theory, the transfer operator approach [137, 158] and some others. Network approach provides another possibility to characterise complex systems [143, 211, 254], which is introduced in details in the following section.

7 Analysis of systems using flow-networks

Could you play a hymn on the downspots?
V.Majakovskij

In this section I present *the method of flow-network construction of time-dependent velocity fields*, generalised from the method for the flow-network for stationary flow systems.
The *structure* of this methodological section is the following. First I introduce several central definitions in Subsection 7.1, then introduce the main idea of the novel flow-networks method in Subsection 7.2, and construction of flow-networks, Subsection 7.3. Finally, in Section 8 I present the analytical and numerical results of the new

method for time-dependent systems with additional parameters.

7.1 Introduction to the flow-networks method

The *concept of flow-networks* is based on the idea to represent a dynamical system, as a complex graph, where links imply information, mass or energy exchange in the system. In Subsection 7.2 I introduce a novel flow-networks method in details for a system, described by linear partial differential equations.
The general *flow-networks method*, presented here, constructs *correlation networks from flows*, which works as follows. Firstly, *linear partial differential equations* are discretized and thus discrete system evolution is defined. Secondly, *stochastic recursive equations* are obtained from the discretized equations, and finally, the *correlation function* between state-vectors is estimated from the stochastic recursive equations.
Let us consider a particular case of partial differential equations, an *advection-diffusion equation (ADE)* with additional forcing and decay terms (on unaccidental choice of ADE I comment in the next sections). In ADE a scalar $T(\vec{x},t)$ (one can think on it as a 'temperature') is transported in a two-dimensional domain by equation:

$$\frac{\partial T(\vec{x},t)}{\partial t} = \kappa\Delta T(\vec{x},t) - \vec{v}(\vec{x},t)\nabla T(\vec{x},t) + F(\vec{x}) - bT(\vec{x},t) + \sqrt{D}\xi(\vec{x},t), \tag{13}$$

where κ is the diffusion coefficient, $\vec{v}(\vec{x},t)$ is the time-dependent velocity field, which is assumed to be incompressible, $F(\vec{x})$ is the forcing, which describes time-independent sources and sinks, $\xi(\vec{x},t)$ is uncorrelated Gaussian white noise with zero mean and correlations $\langle\xi(\vec{x},t)\xi(\vec{y},t')\rangle = \delta(t-t')\delta(\vec{x}-\vec{y})$. The choice of the white noise in Eq. (13) is motivated by [110], where the effect of the random weather excitation on the ocean dynamics is found to be well represented by the white noise. D is the noise intensity and b is a damping parameter, which sets the time-scale at which perturbations are dissipated in the system. Temperature decay and forcing are added to avoid convergence of the scalar distribution to a simple homogeneous equilibrium, and these processes are actually present in geophysically relevant flows. A stochastic term (the white noise) is added to ADE Eq. (13) additionally to the deterministic ones, which results to the stochastic recursive equation. The resulted correlation function, written in the matrix form, is interpreted as a weighted adjacency matrix and analysed using network measures.
In short, the flow-networks method consists of a method of *correlation networks construction* from a given system of partial differential equations, and *analysis of resulted flow-networks* networks using complex networks measures. The complex networks measures are found analytically for advection-diffusion dynamics in Subsection 8.2. This provides new complex networks characteristics of flow regimes for various time-dependent velocity fields, Subsection 8.3.

7.2 Algorithm of flow-networks construction

The algorithm of the flow-network construction for a time-dependent velocity field can be separated into 4 steps:

(Step 1) *Consider first the simplified equation (13), ADE without forcing and temperature decay:*

$$\frac{\partial T(x,t)}{\partial t} = \kappa \Delta T(x,t) - \vec{v}(\vec{x},t)\nabla T(x,t). \tag{14}$$

I discretize Eq. (14) using the Euler scheme for a regular $N \times N$-lattice with spatial resolution Δx and time-interval Δt. The horizontal and vertical components of the velocity field for the lattice point (i,j) at time step $k = t/\Delta t$ are $v_{ij}^x(k)$ and $v_{ij}^y(k)$. This gives:

$$\begin{aligned} &T_{ij}(k+1) = T_{ij}(k) - \\ &\frac{\Delta t}{2\Delta x}[v_{ij}^x(k)T_{i+1j}(k) - v_{ij}^x(k)T_{i-1j}(k) + v_{ij}^y(k)T_{ij+1}(k) - v_{ij}^y(k)T_{ij-1}(k)] + \\ &\frac{\kappa \Delta t}{\Delta x^2}[T_{ij+1}(k) + T_{ij-1}(k) + T_{i+1j}(k) + T_{i-1j}(k) - 4T_{ij}(k)], \end{aligned} \tag{15}$$

where the node's indices are $i,j \in [1,N]$. I use open boundary conditions. The discretisation parameters Δx and Δt should fulfill the Courant-Friedrichs-Lewy (CFL) condition [206] for the stability of the discretisation scheme:

$$\frac{\kappa \Delta t}{\Delta x^2} << 1, \quad \frac{\max(v(x,t))\Delta t}{\Delta x} << 1. \tag{16}$$

The CFL condition is an important issue for discretisation of any type of differential equations, for instance, for Laplace equation or wave equation [250].

(Step 2) *Write Eq. (15) in a matrix form with the one-step transformation operator* $\mathbf{P}(k) = \mathbf{P}(\vec{v}(\vec{x}, k\Delta t))$ for the time step k and the N^2-dimensional state-vector $T(k)$ of components $(T(k)_{\vec{x}})$, denoted as $T_{ij}(k)$, where (i,j) are the lattice coordinates of $\vec{x}$:

$$T(k+1) = \mathbf{P}(k)T(k). \tag{17}$$

The elements of the operator $\mathbf{P}(k)$ are the elements from Eq. (15). Iterating linear stochastic recursive equation (17) leads, for $k \geq k'$, to

$$T(k+1) = \mathbf{M}_{kk'}T(k'), \tag{18}$$

where

$$\mathbf{M}_{kk'} = \mathbf{P}(k)\mathbf{P}(k-1)...\mathbf{P}(k'+1)\mathbf{P}(k') \tag{19}$$

is analogous to the transport matrix defining the flow networks in [215, 228]. Here the transport matrix from Eq. (21) is computed from a discretization of the ADE,

whereas in other works it is computed by the Ulam method (for the description of this method I refer to [85, 88]), that involves the Lagrangian trajectories of particles, but the meaning is the same: it is the matrix that evolves the vector $T(k)$ in time.

(Step 3) *Add the decay term $-bT$ to Eq.(14):*

$$\frac{\partial T}{\partial t} = \kappa \Delta T - \vec{v}(\vec{x},t)\nabla T - bT. \tag{20}$$

and substitute the variable $T(k) = e^{-b\Delta t k}\widetilde{T}(k)$, reducing Eq. (20) to Eq. (14) for $\widetilde{T}(k)$. Therefore the one-step solution from Eq. (17) becomes:

$$T(k+1) = e^{-b\Delta t}\mathbf{P}(k)T(k). \tag{21}$$

The largest eigenvalue (by modulus) of the matrix $\mathbf{P}(k)$ is 1. The new one-step transformation $e^{-b\Delta t}\mathbf{P}(k)$ will have eigenvalues which in modulus are smaller than 1, ensuring that perturbations are damped [76].

(Step 4) *Reintroduce the forcing terms $F(\vec{x}) + \sqrt{D}\xi(\vec{x},t)$ from Eq.(13) into the discretized framework Eq.(15).* This can be done, for example, by integrating them with the Euler method. The one-step solution becomes then

$$T(k+1) = e^{-b\Delta t}\mathbf{P}(k)T(k) + \Delta t F + s\epsilon(k). \tag{22}$$

F is the time independent spatial forcing vector, and $\epsilon(k)$ is, at each time, a vector of independent Gaussian random variables of zero mean and unit variance. These vectors are uncorrelated at different times. One can build correspondence between discrete and continuous noise terms $s\epsilon(k)$ and $\sqrt{D}\xi(k)$ from the stochastic Euler method [250]: the intensity of the discretized noise is $s = \sqrt{D\Delta t/\Delta x^2}$. Iteration of Eq. (22) for $(k+1)$ time steps from the initial condition $T(0)$, gives the time evolution of the scalar distribution vector:

$$T(k+1) = \mathbf{G}_{k0}\ T(0) + \Delta t \sum_{l=0}^{k} \mathbf{G}_{kk+1-l}\ F + s\sum_{l=0}^{k} \mathbf{G}_{kk+1-l}\ \epsilon(k-l)\ , \tag{23}$$

where the propagation matrix is defined (for convenience of notation)

$$\mathbf{G}_{kk'} \equiv e^{-b\Delta t}\mathbf{P}(k)e^{-b\Delta t}\mathbf{P}(k-1)...e^{-b\Delta t}\mathbf{P}(k') = e^{-(k+1-k')b\Delta t}\mathbf{M}_{kk'}\ ,\ k \geq k', \tag{24}$$

additionally for notational convenience propagation matrix $\mathbf{G}_{kk+1}$ is defined as the identity matrix: $\mathbf{G}_{kk+1} \equiv \mathcal{I}$.

7.3 Calculation of correlations for advection-diffusion dynamics

Now correlations, associated with time series from Eq.(23), Subsection 7.2, can be computed. In order to calculate *correlations*, first I calculate the *spatial covariance*. Let us consider the direct product matrix $T(k)T(k)^{\dagger}$ (the superindex † means transpose) whose matrix elements are products of the transported field at different spatial points

$\left(T(k)T(k)^\dagger\right)_{\vec{x}\vec{y}} = T(k)_{\vec{x}}T(k)^\dagger_{\vec{y}}$. Operation $\langle . \rangle$ denotes averaging $T(k)_{\vec{x}}T(k)^\dagger_{\vec{y}}$ over realizations of the noise ϵ in the same operation averaging over the initial condition $T(0)$. Additionally $\langle T(0)\rangle = 0$ is assumed, but this assumption is irrelevant for our results, since the final expressions at long times lose dependence on the initial condition. From the fact that $\langle \epsilon(k)\epsilon(k')\rangle = \mathcal{I}\delta_{kk'}$, the direct product matrix is:

$$\langle T(k+1)T(k+1)^\dagger\rangle = \mathbf{G}_{k0}\langle T(0)T(0)^\dagger\rangle\mathbf{G}^\dagger_{k0} + (\Delta t)^2\sum_{l=0}^{k}\sum_{l'=0}^{k}\mathbf{G}_{kk+1-l}FF^\dagger\mathbf{G}^\dagger_{kk+1-l'} + s^2\sum_{l=0}^{k}\mathbf{G}_{kk+1-l}\mathbf{G}^\dagger_{kk+1-l}\ . \tag{25}$$

The first term in the r.h.s. of Eq. (25) gives the evolution of the initial correlations. Because of the properties of the eigenvalues of $\mathbf{G}_{k0}$, this term will decrease with k and become negligible after a number k of steps such that the corresponding time $k\Delta t$ satisfies $bk\Delta t >> 1$. In the same limit averaging Eq. (23) gives:

$$\langle T(k+1)\rangle = \Delta t\sum_{l=0}^{k}\mathbf{G}_{kk+1-l}F\ ,\ b\Delta tk >> 1\ , \tag{26}$$

so that the second term in the r.h.s. of Eq.(25) is $\langle T(k+1)\rangle\langle T(k+1)^\dagger\rangle$. Combining these facts for $bk\Delta t >> 1$ the spatial covariance of the transported scalar is:

$$\begin{aligned}\mathrm{Cov}(T(k)) &\equiv \left\langle \left(T(k) - \langle T(k)\rangle\right)\left(T(k) - \langle T(k)\rangle\right)^\dagger\right\rangle \\ &= s^2\sum_{l=0}^{k-1}\mathbf{G}_{k-1k-l}\mathbf{G}^\dagger_{k-1k-l}\ .\end{aligned} \tag{27}$$

Hence, expression (27), with Eq.(24) gives the formal relationship between the correlations used to construct climate networks, obtained from the matrix $\mathrm{Cov}(T(k))$, and the transport properties of the flow, which are contained in the flow-network matrix $\mathbf{M}_{kk'}$ and enter into Eq.(27) via Eq.(24). In the following subsection I summarize first analytical results.

7.4 Correlation function for flow-networks

From the covariance matrix, Eq. (27), Subsection 7.3, the Pearson correlation can be calculated. In terms of the matrix elements of the covariance matrix, $(\mathrm{Cov}(T(k)))_{\vec{x}\vec{y}}$, the matrix elements of the Pearson correlation matrix $\mathbf{C}(k)$ are:

$$(\mathbf{C}(k))_{\vec{x}\vec{y}} = \frac{(\mathbf{Cov}(T(k)))_{\vec{x}\vec{y}}}{\sqrt{(\mathbf{Cov}(T(k)))_{\vec{x}\vec{x}}\,(\mathbf{Cov}(T(k)))_{\vec{y}\vec{y}}}}\ . \tag{28}$$

Now the correlation network is constructed from the symmetric and semi-positive definite matrix $\mathbf{C}(k)$. Next matrix $\mathbf{C}(k)$ is thresholded to construct a binary adjacency

matrix $\mathbf{A}(k)$ by the absolute value:

$$\begin{aligned} \mathbf{A}(k)_{\vec{x}\vec{y}} &= 1 \text{ if } |\mathbf{C}(k)_{\vec{x}\vec{y}}| \geqslant \gamma \\ \mathbf{A}(k)_{\vec{x}\vec{y}} &= 0 \text{ if } |\mathbf{C}(k)_{\vec{x}\vec{y}}| < \gamma \end{aligned} \tag{29}$$

Within reasonable limits the threshold value γ below which the correlations are set to zero does not significantly affect the result. In the following the link density is kept constant for different flow-regimes. For the same integration time the link density for different plots is resulted to be the same. The resulting thresholded matrix $\mathbf{A}$(k) is the adjacency matrix of correlation or climate network which is analyzed using network measures. In the following the threshold γ is tuned so that obtained flow-network has prescribed link density.

8 Results of flow-network analysis

8.1 Analytical results

The method of flow-networks has the following analytical implications of our main formula Eq. (27):

- In the framework of the linear ADE dynamics used here, *a time-independent spatial forcing $F(\vec{x})$ has no influence on the covariance matrix, as it is constructed from anomalies with respect to the mean. In the same way, white noise intensity s or D disappears when normalizing the covariance to obtain the Pearson correlation coefficient of Eq.(28).* Thus correlation networks become independent from the forcing terms present in the linear ADE Eq.(13) (although these terms need to be present to sustain the fluctuations from which correlations are computed). The equations (27) and (28) are generalizable for for the case in which a colored noise [154] is used for $\epsilon(k)$. The result is that correlated nodes are the ones affected by perturbations coming from locations within the same correlation length and time of the noise.

- For flow-networks constructed from the transport matrix $\mathbf{M}_{kk'}$ (or $\mathbf{G}_{kk'}$), nodes are connected if there is physical transport between them. *For networks constructed from the correlation, Eq.(28), instead, the presence of the product of two propagators, $\mathbf{G}_{k-1k-l}\mathbf{G}^{\dagger}_{k-1k-l}$, in each term of the sum in Eq.(27) implies that correlations between two nodes will be non-vanishing only if they receive simultaneously (at time k) the effect of fluctuations originated at the same source (at time $k-l$).* This can not happen only by advection, because each Lagrangian deterministic trajectory is unique. Diffusion is needed to spread stochastic perturbations and let them to affect different sites. Thus, links between nodes in correlation networks, constructed from transported quantities, will not represent direct physical transport between them, but the susceptibility for them to be reached by perturbations transported (by advection and diffusion) from the

same origin (and within a time b^{-1} from its birth, because of the exponentially decaying temporal factor in $\mathbf{G}_{kk'}$).

- *Even if for large integration time Eq. (27) involves a large number of terms in the sum, they decrease fast in magnitude, and actually only the ones with l such that $b(k-l)\Delta t < 1$ make a relevant contribution to the covariance or Pearson correlation at time k.* If the velocity field is time-independent, then a transformation operator $\mathbf{P}$ is time-independent. Hence, correlation in the limit for number of time steps $k \to \infty$ can be calculated using geometric series as:

$$\mathbf{C}_\infty = (I - \mathbf{P}\mathbf{P}^T)^{-1}, \tag{30}$$

where I is identical matrix of the same size as matrix $\mathbf{P}$. Additionally, the formula (27) can be generalised for the correlation with a time-lag τ. Then if $\mathbf{P}$ is a transformation matrix for the case of the system with time-independent velocity field, then the formula for the correlation with a time-lag τ, denoted by $\mathbf{C}(\tau)$, is transformed to:

$$\mathbf{C}(\tau)_k = lim_{k\to\infty} \sum_{i=1}^{k} e^{-2bk\Delta t}$$
$$(\sum_{j=0}^{i-1+\tau} (\mathbf{P}\mathbf{P}^T)^{i-j-1} + s^2 \sum_{j=0}^{i-1} (\mathbf{P}\mathbf{P}^T)^{i-j-1}). \tag{31}$$

- *If the velocity field is time-periodic with the period t_p $\vec{v}(\vec{x},t) = \vec{v}(\vec{x}, t-t_p)$,* then there are only t_p various one-step transformation operators $\mathbf{P}(k)$, so that for any k: $\mathbf{P}(k) = \mathbf{P}(k-t_p)$. Thus $\mathbf{G}_{kk-t_p}$ has a special form:

$$\mathbf{G}_{kk-t_p} = e^{-b\Delta t}\mathbf{P}(k)e^{-b\Delta t}\mathbf{P}(k-1)...e^{-b\Delta t}\mathbf{P}(k-t_p) = e^{-t_p b\Delta t}\mathbf{M}_{kk-t_p}. \tag{32}$$

If the total number of time-steps is $k = t_p n$ for n denoting number of periods, then the expression for the covariance can be simplified to the form:

$$\mathrm{Cov}(T(k)) = s^2 \sum_{l=0}^{k-1} \mathbf{G}_{k-1k-l}\mathbf{G}^t_{k-1k-l} \; = s^2(\mathcal{I} + e^{-2b\Delta t}\mathbf{M}_{k\,1}\mathbf{M}^t_{k\,1} + ... \tag{33}$$
$$... + e^{-2t_p b\Delta t}\mathbf{M}_{k\,t_p}\mathbf{M}^t_{k\,t_p} + + e^{-2(t_p+1)b\Delta t}\mathbf{M}_{k\,t_p}\mathbf{M}_{k\,1}(\mathbf{M}_{k\,t_p}\mathbf{M}_{k\,1})^t + ...$$
$$... + e^{-2t_p nb\Delta t}\mathbf{M}^n_{k\,t_p}(\mathbf{M}^n_{k\,t_p})^t).$$

Then for the covariance matrix only t_p matrices $\mathbf{M}_{k\,i}, i \in (1, t_p)$ are needed to be calculated instead of k matrices. The implication of Eq. (34) requires less numerical operations, than for the calculation of the covariance using the formula (27) for flow-networks for the general time-dependent velocity field.

- $\mathrm{Cov}(T(k))$ *from Eq. (27) is a time-dependent matrix, as it depends on* $\mathbf{G}_{kk'}$ *and thus on* $\mathbf{P}(k)$*, which inherits the time-dependence on the velocity field* $\vec{v}(\vec{x},t)$. Because of the temporal averaging implicit in Eq. (27), temporal scales of the velocity field faster than the time scale b^{-1} will be averaged out from $\mathrm{Cov}(T(k))$, but slower time-dependencies will remain and the resulting correlation network will be a temporal network [122]. Therefore b characterizes different "memory" in the system, as it controls how many components will be included in the sum of Eq. (27).

- *In the case without advection (or advection with a constant and homogeneous velocity field* $\vec{v}(x,y,t) = \vec{v}_0$*) Eq.(13) can be solved exactly and the Pearson correlation can be analytically computed.* The resulting network is a fully homogeneous graph in which every node is linked with all neighbor nodes within a correlation length given by $\sqrt{\kappa/b}$. In the presence of non-homogeneous advection, the correlation network from flows becomes inhomogeneous with properties determined by Eq.(27) which encodes, via the propagator $\mathbf{G}_{kk'}$, a non-trivial interplay between advection, diffusion and decay.

The proof of the last implication is the following. For a constant v, the operator $Dif = \kappa\nabla^2$ commutes with the operator $Adv = v\nabla$. Thus the transport matrices P_D, associated with *diffusion*, and P_v, associated with *advection*, for constant transition matrices give simply: $P_D = e^{(tDif)}$ and $P_v = e^{(tAdv)}$ [91], thus matrices P_D and P_v commute. Then the transport matrix for advection-diffusion is simply the product without any additional terms: $P_{(Adv+Dif)} = P_D P_v$. P_D is symmetric and selfadjoint operator, whereas the adjoint of P_v is the same as reversing the sign of v: $(P_v)^\dagger = P_{-v}$. P_v and P_{-v} are inverse matrices (one moves material in one direction and the other in the reverse direction; their product is the identity matrix). Then each product in the sum for the correlation function is of the form

$$G_{(Adv+Dif)}[G_{(Adv+Dif)}]^\dagger = e^{(-bt)} P_D P_v (e^{(-bt)} P_D P_v)^\dagger = \tag{34}$$

$$= e^{(-2bt)} P_D P_v P_{-v} P_D = e^{(-2bt)} P_D^2. \tag{35}$$

The last implication is true due to the properties of matrix exponents. Then, for constant velocity v, the velocity does not contribute to the correlation. It is the same as in the absence of velocity, which can be calculated. After doing the temporal sum to calculate the correlation, the result is a function of $|x - x'|/l$ where l is length where the correlation length is $l = \sqrt{\kappa/b}$. For non-constant velocity the propagators do not commute, and then this argument does not apply.
At the same time, discretisation of Eq. (14) gives discrete operators Dif_{disc} and Adv_{disc}, which are approximations of operators of diffusion and advection correspondingly. Then the "discretisation error", accumulated in the solution for t time-steps $T(t) = e^{(tDif_{disc})} e^{(tAdv_{disc})} T_0$, has the effect on the solution, known as numerical diffusion, which is further discussed in Subsection 8.2.

8.2 Numerical results

The correlation matrix $\mathbf{C}$ is numerically approximated by matrix $\mathbf{C}(k)$ for k time steps, using formulas (27) and (28), analytically derived in Subsection 7.2. The thresholded matrix $\mathbf{C}(k)$ is denoted by $\mathbf{A}(k)$. k time steps are chosen so that the system reaches statistical equilibrium. The network measures are calculated for different values of k time steps Δt, which satisfies condition $bk\Delta t >> 1$.
The main numerical results are *complex networks measures for the flow-networks*, shown in Figs. 30 and 31. Analysis of flow-networks on these figures, more particularly, topological properties depending on dynamical regimes of underlying flow system, is presented in Subsection 8.4, For this in Subsection 8.3 I define various *flow regimes* for the meandering flow [38].

8.3 A model flow setup for flow-networks

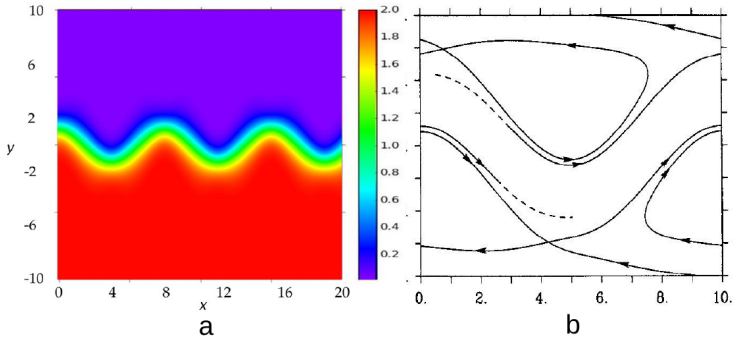

Figure 27: The streamfunction for the velocity field of the meandering flow is constructed for $t = 0$, subplot (a). It describes a jet flowing from left to right, more intense in the central meandering core. The streamfunction is plotted here for $\nu = 0$, and it is the same as for any other value of ν if $t = 0$ or a multiple of the flow period. The meander amplitude is changing in time as: $B(t) = B_0 + \epsilon cos(\omega t + \theta)$ for fixed parameters $B_0 = 1.2$, $c = 0$, $\omega = 0.4$, $\epsilon = 0$, $\theta = \pi/2$. The bounding streamlines for a time-dependent meander for the parameters values on the subplot (b).

To illustrate the use of the analytical formula for correlation, Eq. (28) in Section 7.4, I choose a *meandering flow* model [45, 90, 155, 217, 218] to construct flow-networks. It resembles the simplified velocity structure present in ocean currents such as the Gulf Stream or the Kuro-Shio. The main formulas of flow-networks construction are Eqs.(27) and (28), where I substitute given velocity field. The velocity field $v(x, y, t)$ of a time-dependent meander is obtained from a time-dependent streamfunction, given in [45] as:

$$\Psi(x, y, t) = 1 - \tanh\left[\frac{y - B(t)\cos\left(m(x - ct)\right)}{\left[1 + m^2 B(t)^2 \sin^2\left(m(x - ct)\right)\right]^{\frac{1}{2}}}\right], \tag{36}$$

where m is a wave (meander) number which I set to $2\pi/L_x$, $L_x = 7.5$ and $B(t)$ is the wave amplitude, given by $B(t) = B_0 + \nu\cos(\omega t + \theta)$. Moreover, regions of the velocity field, denoted by Eq. (36), contain flows with more simple structure. Altogether this makes a meandering flow a suitable model to test a novel flow networks method. Considering a prototypical meandering flow allows, to some extent, to generalize results for other spatial structures. Fluctuations of the meandering flow model, denoted in Eq. (36), produce characteristic time-dependent spatial patterns, which also features time-dependency in other flow-models and oceanic flows [45, 255]. A snapshot of the streamfunction in Eq. (36) is plotted in Fig. 27. It describes a jet flowing towards the positive x direction, more intense in the central core region, and meandering in the y direction. The *fixed parameters* are: $B_0 = 1.2$, $c = 0$, $\omega = 0.4$, $\theta = \pi/2$. The types of time-dependent meandering movement:
-the *static meander*: amplitude of fluctuation $\nu = 0$,
-the *oscillating in amplitude* meander with $\nu = 0.7$.
In the first case particle motion in the flow is integrable, whereas in the second chaotic motions arise [45, 155]. From $\Psi(x, y, t)$ the velocity field $\vec{v} = (v^x, v^y)$ is calculated as:

$$v^x(x, y, t) = -\frac{\partial\Psi(x, y, t)}{\partial y} \quad , \quad v^y(x, y, t) = \frac{\partial\Psi(x, y, t)}{\partial x} \ . \tag{37}$$

Then resulted bidimensional velocity field from Eqs. (37) and (36) are substituted to Eq. (28) in order to construct flow-networks.

Flow regimes setup

All in all, I consider two different *flow regimes*, characterized by the various damping parameters: $b = 1$ and $b = 0.05$, introduced in Eq. (13). Interestingly, parameter b corresponds to lifetimes of the perturbations: $b = 1$ "creates" much shorter perturbations ($b^{-1} = 1$) than the time scales of the flow (as given by $2\pi/\omega \approx 15.7$), i.e. shorter "memory" of the system. While for $b = 0.05$ (since lifetimes of the perturbations are much longer: $b^{-1} = 20$) the system has longer "memory". For the flow all parameters are fixed as mentioned above, except the one giving the temporal modulation of the meander amplitude: $\nu = 0$, representing a steady flow or $\nu = 0.7$, giving a time-dependent flow. Therefore the various *meandering regimes* are: (1) static meander: $\nu = 0, c = 0$ (2) amplitude fluctuating of meander along the jet direction at a phase velocity $\nu = 0.7, c = 0$.
I consider the flow domain $x \in [0, 20]$, $y \in [-10, 10]$ with open boundary conditions and discretize it in $N \times N = 120 \times 120$ nodes, so that $\Delta x \approx 0.167$. Time step is $\Delta t = 0.2$. Nominally I take the diffusion coefficient $\kappa = 0.02$, but the numerical diffusion [206], introduced by the discretization Eq. (15), is larger, $\kappa' \approx \Delta x^2/\Delta t = 0.139$. The parameters setup, given above, is identified to be the most convenient in order to demonstrate flow-networks, constructed for various regimes of a time-dependent flow. The findings of this chapter are robust against the change of parameters under assumptions that all conditions of stability of the numerical scheme, e.g. CFL-conditions

from Eq. (16) are fulfilled.
In the following Subsections I estimate complex network measures in order to analyze flow-networks structure: *node degree centrality* is the number of links adjacent to the node, and *node clustering coefficient* is the fraction of triangles actually present through that node with respect to the possible ones, given their neighbors. These measures were introduced in details in [28, 67, 191] and in Chapter II, and since their graph theoretical properties have been extensively studied during the last decades, this makes them suitable measures to test a novel flow-networks method.

8.4 Centrality measures for flow-networks

Comparison between degree and clustering fields

Classical network measures were, for instance, introduced in [67, 191]: the *node degree* is a linear measure by its definition [61]; the *node clustering coefficient* [220]. Clustering coefficient is connected to so-called triadic motifs and pattern formation, known as "motifsâ superfamilies", which are known to be indicators of many transitions happening in the system [13]. These measures together highlight the characteristic patterns of the flow-networks, constructed for basic regimes.
In the following two auxiliary remarks I formulate the main idea behind comparison fields of degree centrality and clustering coefficient values.
Remark 1. Comparison of degree and clustering fields differentiates between various complex networks topologies.
To illustrate the *Remark 1* I construct examples of complex networks, Fig. 28.
Example 1. On Fig. 28(a) a node i is connected to nodes k_q for $q = 1, 2, 3$. Local clustering coefficient of node i helps to differentiate between cases: when neighboring nodes of a node i (nodes $\{k_q\}$ for $q = 1, 2, 3$) are connected to a node i, are also connected with each other or not. On Fig. 28(b) nodes k_1 and k_2 are connected, then the total number of triangles with node i increases, which increases the clustering coefficient of node i. Hence, depending on connectedness of each of the node $\{k_q\}, q = 1, 2, 3$ of network, node i may have both high degree and high clustering coefficient and therefore form a *clique* (group of nodes when all the nodes are connected). Translating the *Remark 1* into language of flow-networks, we get: nodes with high degree and clustering coefficient are those nodes, which are correlated to each other. In details, this implies that the correlation between a state vector $T_i(t)$ and each of state vectors $T_{k_q}(t)$ is higher than a threshold γ of a flow-network.

Remark 2. Degree field (or, in other words, degree sequence) alone does not uniquely specify the network itself. I construct *Example 2*, in order to demonstrate that degree field does not give the whole information about network topology.
Example 2. Two networks on Fig. 29 are non-isomorphic, however they have the same sequence of degree values $(\{deg_i\}_{i\in[1,6]}) = (1, 1, 2, 2, 3, 3)$ for non-numbered nodes.
Hence, degree sequence gives an "average picture" of the whole network, since each

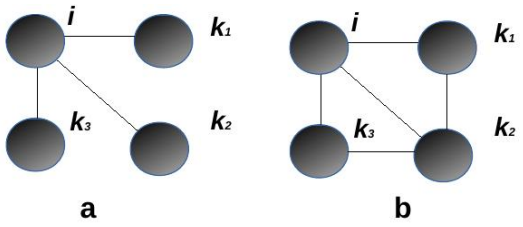

Figure 28: Illustration for *Example 1*: on a subplot (a) a node i has degree 3, the highest degree in the network. At the same time it is connected to nodes k_1, k_2, k_3, which are disconnected from each other. On a subplot (b) node i has degree 3, in addition, nodes k_1, k_2, k_3 are connected to each other, so that node i has high degree and also high clustering coefficient.

degree is a sum of all non-zero entities of an adjacency matrix. Moreover, according to Erdős-Gallai theorem [50], there exists a graph for a given a sequence of degrees $deg_1 \geq deg_2 \geq ... \geq deg_n$ if and only if there exists a simple graph whose nodes have precisely the sequence of degrees with the property: $\sum_{i=1}^{k} deg_i \leq k(k-1) + min\{k, deg_i\}$. Note that local clustering coefficients of the network on Fig. 29(a) equals $(0, 0, 0, 0, 0, 0)$, while network on Fig. 29(b) has a sequence of local clustering coefficients $(0, 0, 0, 1, 1, 1)$. Hence, the remarks and examples above give the intuition behind combining the degree and clustering coefficient fields. Based on this I suggest the main points of flow-networks analysis:

1. Calculate *degree* and *clustering coefficient* network measures for flow-network, using the method from Subsection 7.4. Each flow-network is constructed for parameters defining separate flow regime, as in Subsection 8.3.

2. Compare degree and clustering coefficient fields of flow-networks, which correspond to various flow regimes.

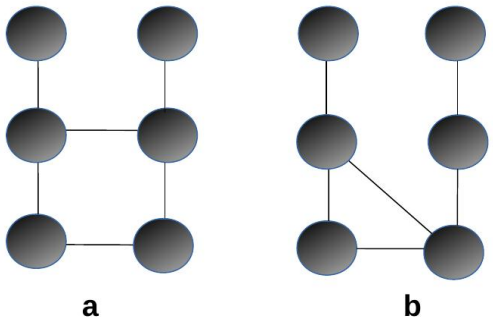

Figure 29: Non-isomorphic network examples for the same degree sequence. Illustration for *Example 2* in Subsection 8.4.

Applications of centrality measures to flow-networks

Here I describe the numerical results of analytic formulas from Subsection 8.4. The adjacency matrix $\mathbf{A}(k)$ of the correlation network is constructed from Eqs. (27) and (28). In the sum in Eq. (27) a number of terms in the sum $k = 314$ for $b = 1$ and $k = 942$ for $b = 0.05$ (which satisfy the condition $bk\Delta t >> 1$) is sufficient to pass the spin-up period in which the initial correlations (the first term in the right-hand-side of Eq.(25)) are still important, and to reach the asymptotic statistical regime. When $\nu = 0$ the flow is static, the streamfunction for it is plotted in Fig. 27, and then the network constructed from $\mathbf{A}(k)$ is also static. When $\nu \neq 0$ the flow, and the correlations and the network is periodic with the period $2\pi/\omega$. For the values used for k, the times $k\Delta t$ correspond to exactly 4 or 12 periods after time $t = 0$ so that at these instants the streamfunction is also the one plotted in Fig. 27. Nevertheless the results below are valid for other parameter values and chosen here for the convenience of numerical calculations. To highlight the spatial structures in the network the threshold γ is fixed such that the link density is 0.075 for the cases with $b = 0.05$, and the link density is 0.003 for $b = 1$. Because of the different values it is not possible directly to compare the absolute values of the network metrics computed at different b. I will be only interested in the spatial patterns. Moreover, although details of the degree and clustering distributions vary, changing the link density in a factor of two does not alter the location of the regions of high and low values of degree and clustering with respect to the ones in Figs. 30 and 31.

The degree of the nodes in the network is plotted in Fig. 30 for the four combination of parameters $\nu = \{0, 0.7\}$ and $b = \{1, 0.05\}$. Fig. 31 displays the corresponding clustering values for each combination of parameters.
In the static case ($\nu = 0$, panels A and B of Figs. 30 and 31) the streamfunction, given in Eq. (36) is constant in time, and plotted in Fig. 27. *Importantly, as expected from Eq.(27) and the discussion above, regions of high degree are not precisely associated with strong currents, and as it was thought before [177].* Nevertheless, when damping rate is fast ($b = 1$), Fig. 30 (A) the general spatial structure of the degree reflects the meandering shape of the flow. *The similarity is stronger between flow and clustering plots, Fig. 31 (A): patches of strong clustering follow the meander structure, with high clustering usually are associated to zones of low degree, and vice versa. This is another new result, which was not observed in the previous studies on correlation networks.*

Surprisingly, the situation completely changes for $b = 0.05$, Figs. 30(B) and Fig. 31(B). Here both degree and clustering fields become nearly homogeneous, with only some weak structure elongated on the horizontal x direction. *The reason is that now many terms corresponding to different times contribute to the sum in Eq. (27), averaging the resulting correlations that lose spatial structure.*
If the temporal dependence of the flow is on, $\nu = 0.7$, little changes are observed. For the case $b = 1$ Fig. 30(C) and Fig. 31(C) this is easy to interpret, since as

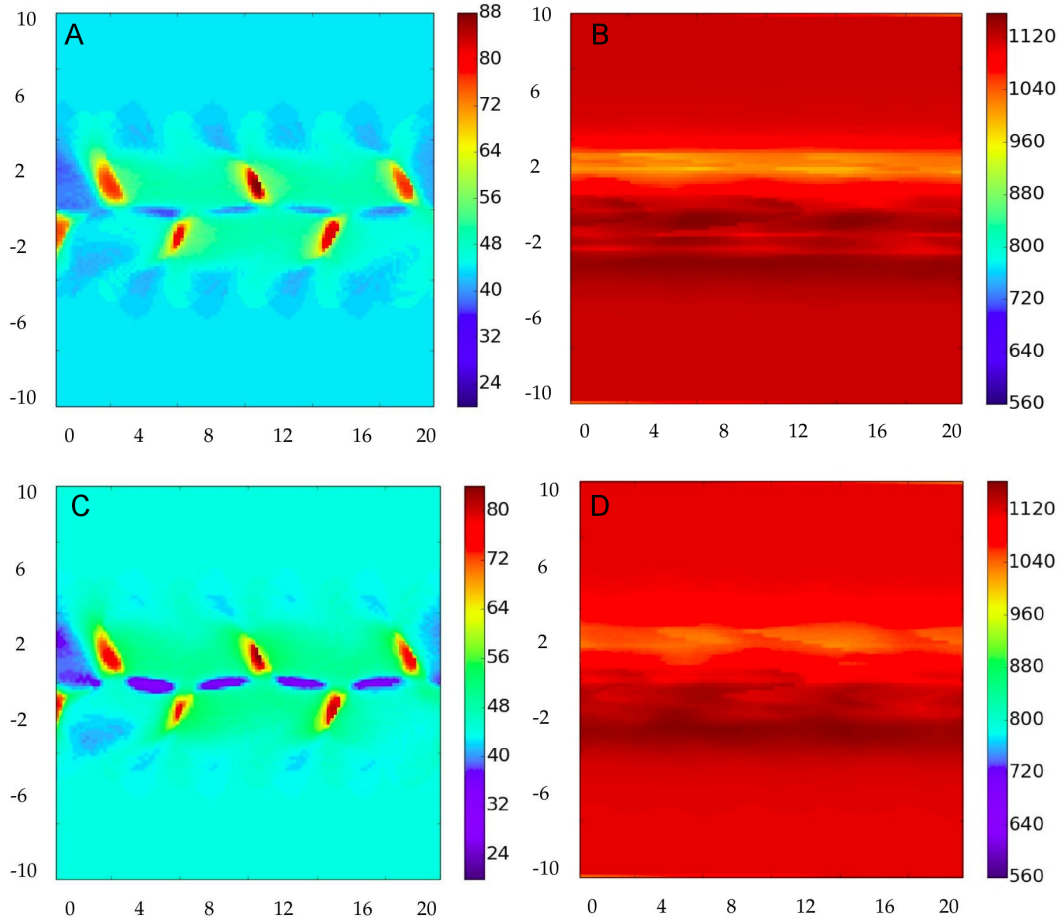

Figure 30: Node degree centrality for the correlation networks constructed for different flows and decay rates. The direction x is horizontal and y is the vertical. Panels A and B display the case of the static flow, $\nu = 0$. C and D are for the amplitude-changing case, $\nu = 0.7$. The dynamic network in this case is plotted at a time after $t = 0$ multiple of the flow period. Then, for all panels the streamfunction at the time plotted is the one shown in Fig. 27. Panels A and C are for the fast decay case $b = 1$, and B and D are for the slow decay, $b = 0.05$, of the transported substance. Other parameters as stated in the text.

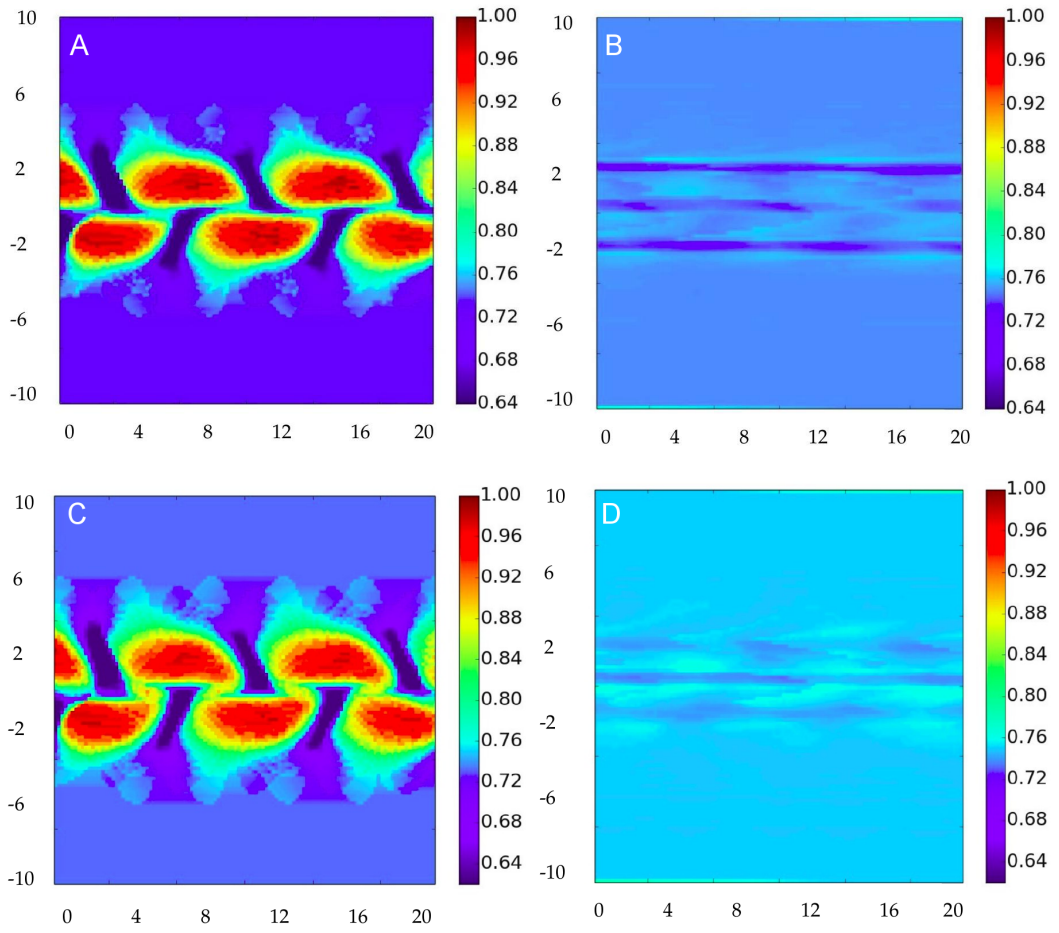

Figure 31: Node clustering coefficient for the correlation networks constructed for various flows and decay rates. Panels are for the same parameters as in Fig. 30.

discussed above only a few terms in the sum in Eq. (27), the ones with $(k-l)b\Delta t < 1$, contribute. For them the flow stays essentially unchanged (remember than the time scale for changes in the slow is $2\pi/\omega \approx 15.7 \gg b^{-1} = 1$). *Thus the results for $\nu = 0.7$ for the case $b = 1$ should be nearly equivalent to the static case.* In fact, only small increases in degree in the central parts and decreases of degree at the maximum are seen in Fig. 30(C) with respect to the static case Fig. 30(A). *Despite the long-time transport properties are rather different in the static and time-dependent case (in particular Lagrangian transport is chaotic at $\nu = 0.7$ [45]) a large damping b restricts the correlations to be influenced only by the short term dynamics, which is similar to the static case.*
Making the decay rate slower ($b = 0.05$), Figs. 30(D) and 31(D) in this dynamic case for $\nu = 0.7$ has also the consequence of homogenizing the spatial structure, in a manner similar to that of the static case. *The structure for the case $\nu = 0.7$ is slightly more homogeneous than for $\nu = 0$, because of the additional mixing associated to the chaotic dynamics.*

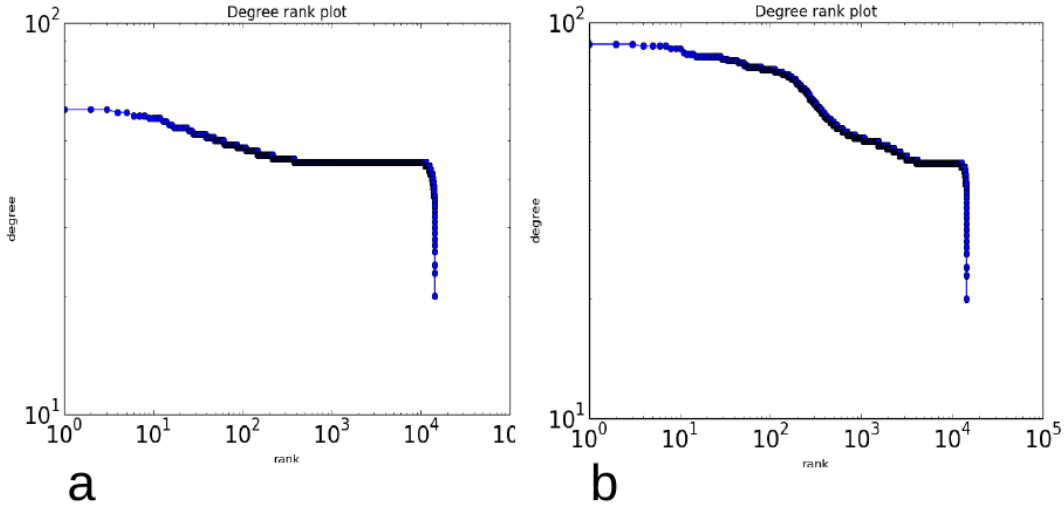

Figure 32: Degree distributions (non-normalized) for the flow-networks for static meander (a) and moving meander (b) for damping parameter $b = 1$. Link density is 0.0035%, other parameters are defined in Subsection 8.2. Degree values are on the vertical axis, nodes' rank is on the horizontal axis.

Global network measures for flow-networks

Global network measures are useful indicators for highlighting local and global properties of the system, as it was shown in Chapter II. I applied global modularity-measure to flow-networks, constructed for different ν values, corresponding to different mixing regimes [45] of the system. *Global modularity measure (GMM)* quantifies the amount of nodes grouped together into modules. In other words, GMM is strength of

division of a network into modules (also called groups, clusters or communities).
First I constructed flow-networks for each parameter values ν from the interval $[0, 1]$. GMM was calculated for each of these flow-networks. A curve on Fig. 33 changes its slope for amplitude fluctuation value ν such that critical value is $\nu_c \approx 0.3$, which gives a signal to some structural reformation of flow-networks as well. For various threshold values one observes a transition in the curve slope around $\nu_c \approx 0.3$. In Fig. 33 the GMM for flow-networks is demonstrated for flow-networks with link density of 0.0035 and integration time $T = 314$ and damping parameter $b = 1$.
Another global network measure is shown in Fig.32. I show degree distributions for various types of meander variability: Fig.32(a) is constructed for a static meander, where most of the links have the same degree, and Fig.32(b) for a moving meander, where the variation of nodes' degree is broader, than for a static case.
Further in Section 9 I discuss the novel flow-networks in respect to other flow-networks methods.

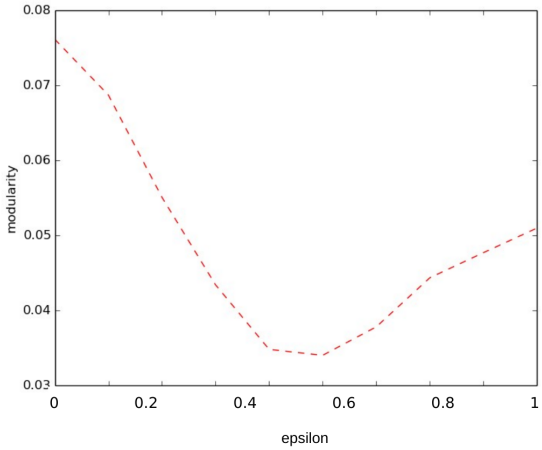

Figure 33: Global modularity measure for flow-networks constructed for various values of parameter ν, which defines the mixing regime. Value of ν_c corresponds to transition between two various mixing flow regimes.

9 Discussions

Flow-networks from Eulerian and Lagrangian perspectives

Previously I demonstrated the applications of the flow-networks method to the model flows from Eulerian perspective. Here I describe the applications of flow-network techniques to characterize mixing features of the fluid using Eulerian and Lagrangian approaches.
In order to compare the construction methods of Lagrangian flow-networks [228] and advection-diffusion flow-networks [147, 254] one can look at the difference in the one-step matrices construction, links properties for each of the network type,

Table 3: Types of flow-networks

Networks	**Lagrangian flow-networks**	**Correlation flow-networks**
Perspectives	Lagrangian flow perspective	Eulerian flow perspective
Construction	Ulam's method [215] propagation matrix estimator	Discrete method of flow-network construction [147, 254]
Properties	Advective properties of the system (data or model)	Advective, diffusive components, noise, dissipation rate of the system
Measures	Degree, FTLE (Lyapunov exponent) MPP-betweenness [228]	Degree, clustering coefficient [254]. Anisotropy [147].

interpretation of the network measures, Table 3. Both approaches of the flow-network construction are based on construction of the one-step matrices: the transition matrix for *Lagrangian flow-network* is built using the Ulam's method, while the transformation matrix for the *correlation flow-network* is built using the discretisation of the advection-diffusion equation (13). The Ulam's method [88] for construction of transition matrix is presented in Fig. 34. In the case of *Lagrangian networks* the transition matrix is the result of the drift of the particles, while the transformation matrix for *correlation network* is defined as the approximated solution of the ADE for fixed in space grid-points. The meaning of the transport matrices for both approaches is basically the same: it is the matrix that evolves the state vector $T(k)$ in time.

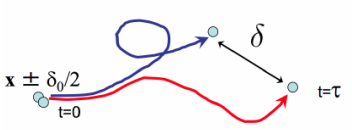

Figure 34: The Lagrangian network is constructed using advective properties of flows. The adjacency matrix is estimated using so-called Ulam's method [88, 228].

One can directly relate the network measures to the flow properties, such as dispersion rate and such as the finite time Lyapunov exponent [228, 229]. Here I apply and analyze network measures for the correlation flow-networks in order to study the underlying dynamics.

Mixing properties of time-dependent flow

Here I additionally describe some mixing properties of flows. It has been shown that transitions to the large scale chaos between regimes are observed in a time-dependent flow for its certain parameter values [217]. Degree and clustering measures for flow-

networks for bidimensional meandering velocity fields before and after transition to Lagrangian chaos are shown in Fig. 30 and 31. It is known that for a three-dimensional system it is not necessary to consider the time-dependent flow in order to obtain the large scale chaos. The typical example is ABC-flow [9]. The proof of transition to the Lagrangian large scale chaos was shown using Melnikov method [105] developed in 1963.
Below I briefly give the idea of *the proof for the Melnikov method*, which gives the intuition to understanding of emergence of Lagrangian chaos [217]. The idea is to use the nonlinear stability of dynamical system and to show that the trajectories of the particles in the vector field start intersecting in the presence of the small time periodic amplitude perturbations (the trajectories in the classic Melnikov method are considered in the phase space, while here the trajectories are in the physical two dimensional space). Let us consider particular flow of meander, Fig. 27. The flow is divided into several regions with different regimes by two pairs of separatrices that connect the stagnation points that occur above meander crests and below troughs. In the presence of the variability included in the streamfunction, the boundaries may break, and exchange between regimes occur. Suppose that the flow varies only slightly from the basic meander, Eq. (36). Then let us consider a fluid particle in the trough of a meander very close to the boundary between the jet and recirculation regimes in the basic jet. The particle will follow the boundary closely as the time progresses, since motion across the streamlines of the basic flow will be driven only by the small variability. Then it will approach the stagnation point above the downstream crest and continue in one of the circulation directions (up- or down-stream). Thereby this determines the following "final regime". The regimes of origin may be determined by following backward in time past the stagnation point above the upstream crest. Since the destination of particles depends in a continuous way upon their initial positions, instantaneous boundaries will exist that separate particles according to their initial and final regimes. In the time-dependent flow, these boundaries no longer need to correspond to streamlines, and therefore they may even cross each other, which makes flow trajectories to become "chaotic". For more details on Melnikov method I refer to [105]. Furthermore I conclude with the numerical and analytical results of the novel flow-networks method.

10 Conclusions and Future prospectives

Herr, es ist Zeit...

Rainer Maria Rilke

To conclude, the main results of Chapter III are the following:

- 1. In this chapter I presented the novel method of flow-networks. It closes a gap in the *theoretical understanding of the relationship between topological networks constructed from correlation functions and the underlying dynamics* of the fluid transport. *The analytic formula of correlation matrix for advection-diffusion*

dynamics has been attained for the time-dependent underlying system, Eqs. (27) and (28). This formula gives the formal relationship between correlations used to construct correlation networks, obtained from the covariance matrix, and the transport properties of the flow, which are contained in the flow-network matrix. The novelty of the results is that the flow-networks method [254] is extended for time-dependent non-autonomous flow systems. In addition to this, the effects from new parameters, such as forcing and damping Eq. (13), were analysed using flow-networks method, which allows to discover new effects of memory in the system, on the correlation network from Eq. (27).

- 2. *This novel method constructs correlation networks directly from flow fields, Subsections 8.1 and 8.2. The results of flow-networks analysis showed that Pearson correlation is used to establish links between nodes. Surprisingly, correlation networks are indifferent to steady sources and sinks of the transported substances, introduced in Eq. (13)*: the normalization in derived Eq. (28) eliminates the dependence on constant forcing and noise intensity. This implication is strictly valid only for the linear ADE dynamics in Eq. (13) and will not apply to dynamics involving nonlinear processes (such as plankton dynamics, vorticity). In geophysical contexts, one can not look into climate networks for information about these processes.

- 3. *It has been found that the relationship between the correlation network, constructed from* $\mathbf{C}(k)$*, and the underlying flow transport network* (characterized by the matrices $\mathbf{M}_{kk'}$ or $\mathbf{G}_{kk'}$) is not direct. It is explained by the fact that the correlation expression involves a sum over time, and each term involves the product of two propagators, meaning that correlated nodes are not the ones connected by the flow, but the ones affected within a time b^{-1} by perturbations coming from a common origin. The analytical results, Subsection 8.1, are affected when a colored correlated noise (e.g. from the common driver) is used for $\epsilon(k)$: The result is that correlated nodes are the ones affected by perturbations coming from locations within the same correlation length and time of the noise. In consequence, patterns of degree and clustering measures are related to flow patterns in a rather indirect way, as Figs. 30 and 31 confirm. Note that this result relies strongly on considering the *equal-time correlation*. In cases in which a *time-lagged correlation* is used [177, 269, 275] the resulting network would be more associated to fluid transport occurring between nodes during the selected temporal lag.

- 4. *The novel method of flow-networks [254] together with other methods of network construction from flow systems [177, 228] enables to detect complementary properties of the flow*, such as to tackle the differences of time-dependent flow regimes. This fact is important from perspective of the dynamical systems analysis and the possible applications to the geophysically relevant flows. There are some advantages and disadvantages of the network techniques in comparison with other existing methods, nevertheless networks are a prominent tool to

visualise dynamical systems. The advantage of presented method is that it is applicable for different types of dynamical systems, e.g. systems defined by parabolic type of differential equation and some others.

- 5. *General relationships between correlation networks and flow-networks have been found. This allows to overcome some restrictions of previous approaches [147, 177, 211].* From the numerical results presented here it is seen that one of the *parameters having the largest impact on the network topology, in fact, more than the flow geometry or temporal variability, is the characteristic time scale of perturbation damping (here represented by the decay rate b).* This important parameter would then have to be taken into account when investigating the structure of climate networks constructed from observed or analyzed data.

Open problems.

The new techniques can seduce to use them without any connection to physical models or data type. Therefore it is important to mention that the novel flow-networks framework, introduced in this chapter, has its limitations. For instance, it may not be applicable when nonlinear measures of statistical dependence, such as mutual information, information transfer [58, 59] or event synchronization [161, 208] replace the correlation function. Also, our flow-networks analysis is restricted to the ADE dynamics implemented by Eq. (13), which considers only material transport. Our conclusions may not apply to climate networks constructed from variables involving wave propagation (such as Kelvin or Rossby waves), such as sea surface height or geopotential [6]. Moreover, it cannot be used for continuous systems where the croase-graining cannot be applied. The flow-networks method can, in principle, be used for some other transport process [122, 167, 239], for studying diffusion [187, 238]. Moreover, the flow-network method allows to study systems with external heating at a subset of the nodes, systems where some interactions are artificially lowered. The nearest further steps are:
1. *Investigation of flow-networks, using other network measures,* such as the shortest path betweenness (for various cutoff values) [83]. Application of global measures, such as the ones, presented in [189, 253], Chapter II, is as well the next step for the flow-networks matrices.
2. *Spectral analysis of correlation and transformation matrices*, including sensitivity analysis of correlation matrices depending on transformation matrices, which is closely related to so-called inverse problem and the perturbation theory of linear operators [129]. Moreover, it seems to be another open question, *to investigate, how structural properties of the correlation matrix depend on a spectral transformation matrix.*
3. *Identification of the most influential nodes in correlation network is connected to so-called model reduction problem* [141, 248]. Inference of nodes in correlation network plays an important role for the reduction of model dimension.
4. *The relation between flow-networks [254], and Lagrangian flow-networks approaches [108, 228],* Table 3 provides potential sources for the future work on flow-networks.

Dynamics on networks

11 Introduction

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Il n'a pas repondu a mes questions...
"Small prince", Antoine de Saint-Exupery

Recently an extensive and detailed graph theoretical analysis of networks with applications to neurobiology, climate and power grids has been performed, and has been particularly discussed in Chapters II and III. A particular example of a complex system is the Earth evolution which cannot be described without a "human factor" anymore [164, 223]. Such a system needs to be considered in coexistence with other components. Recently the concept of planetary boundaries [241] has been introduced, where different components of the Earth system are considered together in co-called *co-evolution*. Co-evolutionary modeling approaches aim at incorporating the complex dynamics of society into the description of natural systems in order to obtain a more holistic picture of the world-earth system. As our world becomes increasingly connected through the use of communication and transportation systems, an understanding of how these connecting networks evolve in time plays an important role.
As an attempt to understand some mechanisms of the complex systems, models on networks with dynamically changing parameters (*graph dynamical systems* or *dynamical network models*) have been mathematically described in [160] and later on further designed in [5, 148]. The nodes of a *dynamical network* (DN) are individual dynamical systems which are coupled through static links. Moreover the network topology can evolve dynamically in time. As the result, combination of dynamics *on* networks and dynamics *of* networks yields a particular class of the dynamical networks, so-called *adaptive network models* [104]. Another class of dynamical networks are *discrete state network models*, where a state of each node is defined by a discrete function evolving in time. An illustrative example of such a model, where each node has a discrete state, is shown in Fig. 35. Studies of analytical and numerical solutions for DN models become a topical issue in natural science [10, 221]. DN models have been successfully studied using graph theoretical approaches, algebraic groups properties of graphs, probability theory and Markov chains [109, 263]. One such approach was explicitly demonstrated in [84], where a network is defined by the transformation matrix of a Markov chain. In particular, directed graphs can be interpreted in the sense that events are represented by nodes of the

graph, and a directed line from one node to another indicates a positive probability of direct succession of these two events. Several of the concepts listed above have been successfully applied to describe a broad spectrum of various types of DN. However it is hard to develop a general theoretical framework for investigation of analytical solutions for DN models since they have structural differences.

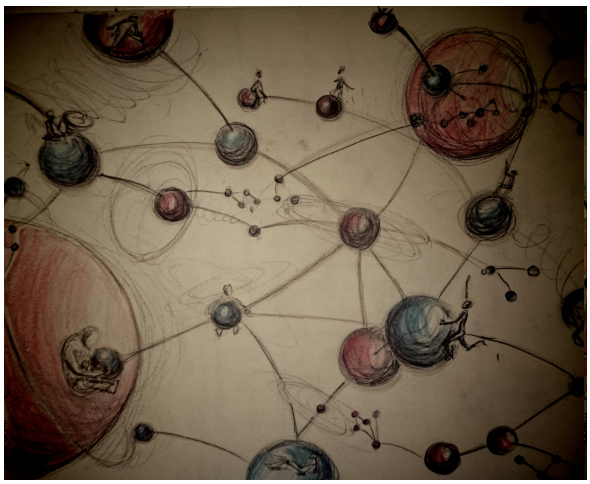

Figure 35: Illustrative example of a dynamical network model: nodes represent humans (or separate communities). Links between nodes correspond to connections between humans, nodes size represents status of a human (HOpS model). Links and states of the nodes can evolve in time.

Here I developed a new conceptual, stochastic **H**eterogeneous **Op**inion-**S**tatus model (HOpS model), presented in details in Section 12. The *HOpS model* admits to identify the main attributes of dynamics on networks and to study analytically the relation between topological network properties and processes taking place on a network. Another key point of the *HOpS model* is the possibility to study network dynamics via the novel parameter of heterogeneity. I show that not only clear topological network properties, such as node degree, but also the nodes' status distribution play an important role in so-called opinion spreading and information diffusion on a network, Subsection 12.2. Furthermore, in Section 13 I propose an analytical method to study DN models demonstrating it on the *HOpS model* on networks with regular topologies. The analytic solutions are also extended by the numerical results from Subsection 13.4.

11.1 Motivation

The process of "spreading out" of a substance is widely used in physics (particle diffusion), chemistry, sociology and others [171, 238]. *Diffusion* is a fundamental transport mechanism with countless examples in nature [18, 96, 244], which leave many open fundamental questions about diffusion processes [17]. The molecular nature of homogeneous diffusion was understood using new approach of Einstein to a random

walk [74]. The study of random walks on different structures such as regular lattices or, for instance, particularly, Cayley graphs [136], allows to understand how certain dynamical processes on networks take place, for instance, energy transfer, chemical reactions and transport problems. Moreover a variety of interesting mathematical problems arise from these studies [231]. Spatial aspects of diffusion and advection processes were recently studied using a flow-networks approach presented in [147, 215, 254], and were discussed in details in Chapter III. Flow-networks are constructed from a discretisation of the advection-diffusion equation on regular grids, which helps to bridge the gap between the dynamics of the system and the topology of the corresponding correlation network. While the purpose of *functional correlation networks* (FCN) is to study data time-series or a dynamical systems from obtained topological properties of FCN, the purpose of so-called *dynamical networks* is to study processes on networks with certain "prescribed" topology, which can be in addition coupled with dynamics on the network [125]. It is clear that the combination (or in other words, adaptation) of non-trivial network topologies and dynamical processes on a network can produce rich dynamics. In many recent works on opinion and coalition formation [10, 103, 221] *dynamical adaptive networks* were used as a prominent tool to analyze complex systems. One can define a variety of DN models on less regular networks, such as small-world networks and many others. Several statistical physics concepts were introduced to describe adaptive dynamics [44], which can also be applied to study social collective behavior. It is not necessary to justify that opinion formation processes play an important role in many aspects of our life [111, 112]. The last years have seen a clear rise of interest in collective phenomena emerging from the interactions between individuals in social structures. Typically, society structure is represented as a network in mathematical approaches to this problem, where a link determines the connection between nodes, see Fig. 37. The ubiquitous real-world examples demonstrate, why it is important to model the information spread processes using DN models defined on networks:

Example 1. Person T lives in country A. T is interacting every day with many people, but only from country A. After some time person T gets a letter from another person from country B with some information about himself (about person T). How is this possible? The reason is that some friend of person T, living in country A, traveled to country B and spread the information about person T to people from country B.

Example 2. One user with a few connections in some internet network wrote some news which were highlighted ("liked") by some "big hub" user in this social network. As a consequence, the news from this "small user" are started to be spread by many other users, Fig. 36.

Nowadays thanks to the advantage of telecommunications, huge amount of data opens great opportunity to understand processes in society and estimate models of them, which was not possible before. But even before the accessibility of such data it was possible analytically to estimate cognitive properties of society. For instance, the sociologists P.Killworth and R.Dunbar defined and estimated a limit to the number of people with whom one can maintain stable social relationships, the so-called, Dunbar's number [72]. Development of social models including accessible applications

to studies of *"flows"* of opinion in society, riot behavior, innovation, strikes, voting and migration, have been extensively deliberated during the last decades [98, 101, 113]. From the series of seminal works it became clear that opinion "flows" are mainly governed by the "network hubs", yet in [132] it was found that node degree is not the only characteristic of the "node importance". Instead, the most efficient spreaders are those located within the core of the network [226]. There is a number of examples demonstrating that information propagation can be represented as *"flows"* or "opinion waves". An illustrative example of such flows is the circulation of ideas among articles through citation network [146, 153].

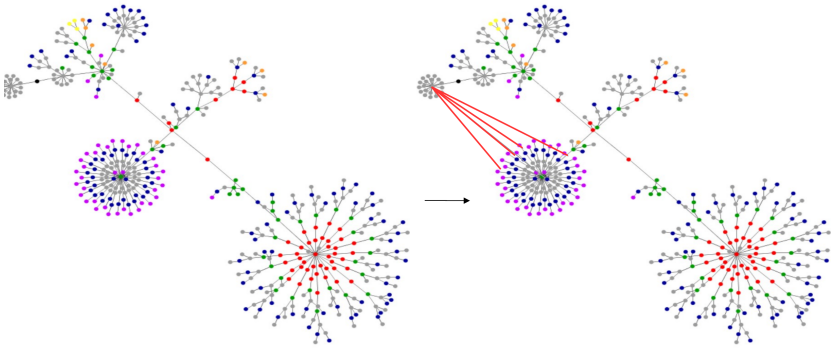

Figure 36: Evolving twitter-network at time step t (a), and at time step $t + m$ (b), when several edges are added in the network.

Let us assume that each person is represented as a node in a network, which represents society. Before to come to the main research questions of this chapter, let us compare the information spread model with a disease contagion model [152], since the opinion spread could also be understood as a special type of contagion. In a recent work [214] the issue on difference between complex and simple contagion models has been addressed: in simple contagion models (SIR models) the most influential nodes are typically the nodes with high degree and low clustering, while in complex contagion models the most influential nodes are typically characterized by low degree and high clustering. The main aspects, which differentiate various types of *contagion spread mechanisms*, can be conditionally separated into:

- *The mechanism of spreading*, which is determined by properties of spreading, the stochastic or deterministic character of the information spreading, etc.
- *The mechanism of node state change*, which determines how each node changes its state with dynamics on the network: for example, resistance to change its current state.

In the series of recent works [93, 221] it has been found that disease spread is more likely to be homogeneous among groups and depends mostly on the properties of the nodes, but on the other hand, the speed of opinion circulation strongly depends on the social group properties and society structure [4, 23, 150, 272]. Ties strength

is important for social contagion which can be modeled as the status difference of the nodes [101], one of possible examples is a weighted voter model [131]. This gave motivation to design a particular novel type of DN model, the HOpS model, where a heterogeneity parameter plays an important role in dynamics of the model.

11.2 Research questions: graph dynamical systems

Ultimately, in this chapter I focus on the *following research questions* about DN graphs:

1. What are the conditions for a dynamical network model to come to an equilibrium? What is the speed of convergence towards the equilibrium? Is the equilibrium state unique, Fig. 38? How to characterise properties of phase space of dynamical network model analytically?

2. Is it possible to estimate the model evolution on a certain network topology without numerical simulations, for instance, using the transformation operator approach?

3. How is the underlying network topology reflected in the model's dynamics? Are there any network topologies for which the model can be completely analysed? What are the effects of heterogeneous spread of opinion on the network?

All in all, I examine behavior of the HOpS model looking at these questions. In order to complete the research questions first I formulate a brief classification of DN models, Section 12, and methods overview to the existing methods in Subsection 12.1.

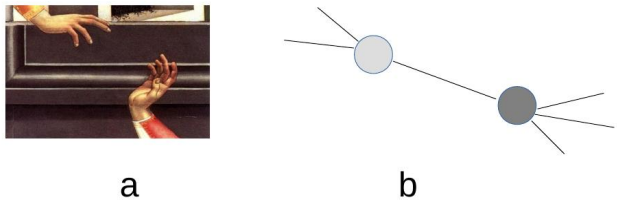

Figure 37: Connections between people have been topic of discussions besides artists and scientists since the ancient times. The fragment from Botticelli's painting (a) illustrates a human interaction (b).

12 Dynamical network models classification

Wachet auf, ruft uns die Stimme
BWV 140, J.Bach

Any decent classification allows to find out, what has been done in a field and what remains undiscovered, however, recently new DN models on networks have not be structurally classified. Social, economic and biological networks can be modeled as

dynamical network models or agent-based models [18, 32, 36, 37, 121], where each node has the possibility to change its state. Looking into real-world complex networks one can find many instances of networks whose states and topologies coevolve, i.e. they interact with each other and keep changing often over the same time scales, due to the system dynamics [114, 222]. Depending on a type of a process being modeled, nodes of the dynamical network can have *discrete and/or continuous* evolving states, *adaptive and/or non-adaptive* strategies of evolution, and simultaneous switch of states of all nodes or a switch in a randomly picked node. Following such an approach divides properties of DN models of into classes, shown in Fig. 39.

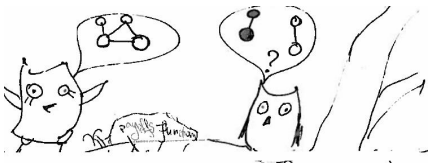

Figure 38: Illustration of dynamical network model with different types of equilibrium states: total consensus (on the left) or separation into smaller groups (on the right), and in each of them consensus is reached.

Graph theoretical notations for dynamical network models

Let us describe the classification of DN models using graph theoretical notations, as it has been done in Chapter II for evolving networks. Let us denote a dynamical network model on an underlying graph $G = G(V, E)$ as $G(V, C(t), E)$, or simply as $G(t)$, where sets V and E are sets of nodes and edges correspondingly, number of nodes $|V| = N$, $C(t)$ is a set of nodes' states at time step t, where a state of node i is denoted by $c_i(t), i \in [1, N]$, where $c_i(t)$ takes values from a fixed set Q. For simplicity we fix sets V, E and consider only finite subsets of integer numbers, which can have possible values. Let us denote a function F, acting on a set of nodes' states. In fact, this can be also written in matrix notations. Let us denote $C(t)$ as a vector state of enumerated nodes's states at time t and F is a matrix, defining transformations of nodes' states (the exact form of this is given in Subsection 13.1). Then the evolution of dynamical network can be written according to a formula $FC(t) = C(t + 1)$, or in other words the evolution of a whole DN model on a static network topology can be written as:

$$G(V, C(t + 1), E) = F(G(V, C(t), E)) = G(V, F(C(t)), E). \quad (38)$$

Generally, each node of a DN model $G(t)$ may have several types of characteristics, instead of only one type $c_i(t)$. This can be encoded using additional set of nodes' states $\{C^1, ...C^k\}$, each set C^j for j^{th} type of nodes' characteristics, so that then a DN model would be denoted as $G(t) = G(V, C^1(t), ...C^k(t), E)$.
As an *example*, let us consider an evolving DN model $G(t) = G(V, C(t), E)$ with fixed set of nodes, set of edges and boolean set of nodes' states $Q = \{0, 1\}$. Let a

deterministic rule of $G(t)$ be: at each time step t a state of each node $c_i(t), i \in [1, N]$ is changing its value to opposite value: $0 \to 1, 1 \to 0$. Then a deterministic function F determining state of each node is $F(c_i(t+1)) = (c_i(t)+1) mod 2$.
Let us consider properties of functions, acting on set of nodes' states. Let us assume that function F_i acts on some subnetwork $G_i \subseteq G$. Assume, that a function F is a composition of functions: $F = F_1 F_2 ... F_n$. Notably, function F_i does not necessarily commutate with function F_j, acting on another subnetwork $G_j \subseteq G$. Therefore, it is essential to distinguish the order in which functions are applied: $F_i F_j(G)$ and $F_j F_i(G)$.
Hence, characteristics of function F, such as in Eq. (38), typify the evolution of DN models. In particular, F may act on a set of nodes' states depending on edges evolution or independently on edges evolution (adaptive/non-adaptive networks); values of $C(t)$ may evolve in discrete or continuous time; F may act on the whole network, or be applied to separate subgraphs of a network using synchronous or asynchronous update mechanism.
The model is evolving until a DN model reaches a final configuration, which can be either a consensus for the whole network or consensus, reached in disconnected small subnetworks, Fig. 38. Furthermore, in Subsection 13.1 I look at problems of DN models from another perspective of so-called sequential dynamical systems (SDSs) [20, 160], which helps to describe a discrete phase space of DN models.

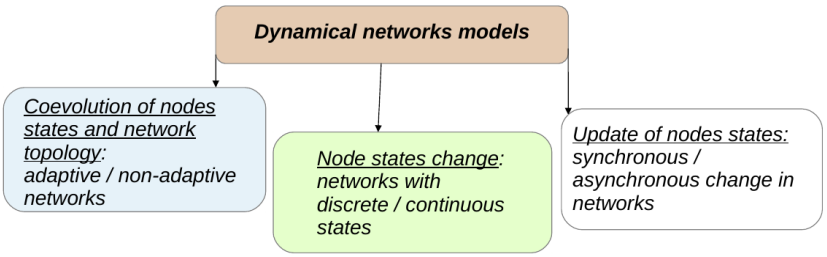

Figure 39: Properties of dynamical networks models are divided into several classes depending on characteristics of functions, acting on node states, type of coevolution between nodes states and network topology.

12.1 Techniques to describe dynamical networks models

> Swiftly arose and spread around me the peace
> and joy and knowledge that pass all the art
> and argument of the earth
>
> Whalt Whitman

There exist various methods to study DN models, such as transfer operators approach, numerical approach to run models on ensembles of random topological graphs. Various random network models provide an efficient laboratory for testing various

collective phenomena in statistical physics of complex systems, and are, on the other hand, tightly linked to statistical, topological properties of random matrices, for instance, vertex degree distribution, clustering coefficients, "small world" structure and spectra of adjacency matrices [13, 22, 139].
In order to illustrate one quite famous approach for studying dynamical system with discrete phase space let us consider the discrete transfer operators approach to study Moran model [180]. At each time step a random individual of one of two types A or B is chosen for reproduction and a random individual is chosen for death; thus ensuring that the population size remains the same (the number of individuals is conserved). Here Moran process can be used to analyze variety-increasing processes such as mutation, as well as variety-reducing effects such as genetic drift and natural selection. This process describes the probabilistic dynamics in a population of finite constant size N in which two species A and B are competing for dominance. The main idea of the *transition matrix method* is as follows. Let us consider system with discrete number of states, numerated by finite numbers $0 \leqslant i \leqslant N$. A phase space (or state space) is one-dimensional and discrete set $\{1, ..., N\}$, where N is the number of possible states (in other words, the cardinality of the phase space). The so-called transfer operator is thus represented by an $N \times N$ transition matrix P acting on a vector of $\mathbb{R}^n$. A *transition matrix* P has entries $P_{i,j}$, which denote the probability to go from state i to state j. To understand the formulas for the transition probabilities one has to look at the definition of the process which states, that always one individual will be chosen for reproduction and one is chosen for death, i.e. $P_{i,j} \in [0, 1]$. Once all A individuals have died out, they will never be reintroduced into the population since the process does not model mutations and thus $P_{1,1} = 1$. For the same reason the population of A individuals will always stay N once they have reached that number and taken over the population and thus $P_{N,N} = 1$. Then states 1 and N are called *absorbing* while the states $2, ..., N - 1$ are called *transient*. Analysing properties of transformation matrices, one can study possible states of the system.
Further I introduce the method based on analysis of transformation matrices, illustrating it on the Heterogeneous Opinion Status model (HOpS), Section 12.2.

12.2 Heterogeneous Opinion Status (HOpS) model setup

> Absolutely white, as the absolutely black
> seems to be some defect of perception...
>
> J.Oruell

A novel dynamical network model, **H**eterogeneous **Op**inion **S**tatus (HOpS) model has the following properties, which allows to demonstrate analytical methods to characterise dynamics on networks.
The HOpS model setup.
Let us consider the network where each node i has two variables: status and opinion. Status is fixed and is denoted by a finite number. Each node i has opinion $op_i \in \{0, 1\}$, 0 is white, 1 is black, Fig. 40, and changes according to a stochastic rule representing

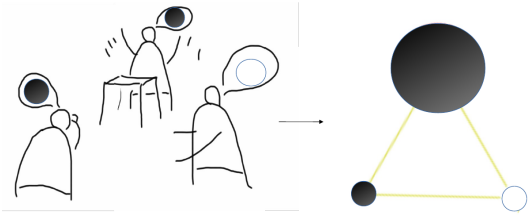

Figure 40: **Representation of the group of people with two types of opinions** marked in black and white colors (a). Heterogeneous Opinion Status (HOpS) model representation of the group from three people (b), where the node status is represented by the node size.

Table 4: Algorithm for the time step of the HOpS model

Phase of time step	**Characteristics of each phase**
1.	Randomly choose an *active node* i from the network, Fig. 41.
2.	Randomly choose *one neighbor* of the active node i - node j.
3.	Change the *opinion of an active node* i to the opinion of node j with probability $p = 0.5 \tanh \sigma \lvert st_j - st_i \rvert + 0.5$.
4.	Go to 1., iterating the whole time-step until an equilibrium state is reached.

the imitation of opinion.

Definition. Node i is called an *active node* in the HOpS model, if node i is randomly chosen at time-step t with its random neighbor j and then active node i is changing its opinion $op_i(t)$ to opinion of its neighbor $op_j(t)$ with a fixed probability, dependent on the difference in statuses.

The algorithm of the *HOpS model time step* is given in Table 4.

Let us now formulate the HOpS model in notations from Subsection 12. The HOpS model is defined on a fixed graph $G(V, E)$ with a changing set of nodes opinions, denoted by $Op(t) = (\{op_i(t)\}_{i\in[1,N]})$ and nodes statuses $St = (\{st_i\}_{i\in[1,N]})$. In other words, the HOpS model can be written as $G(V, Op(t), St, E)$.

The HOpS model input control parameters are:

1) a fixed set of *nodes statuses* St;

2) a distribution of *initial nodes opinions* at time step $t = 0$ $Op(0)$, (this is discussed in details in Subsection 13.1, and moreover, this, on the first place, means that the system is non-ergodic);

3) a fixed underlying network *topology* $G(V, E)$. Moreover, a parameter σ of a time step influences the HOpS dynamics, its role is discussed separately.

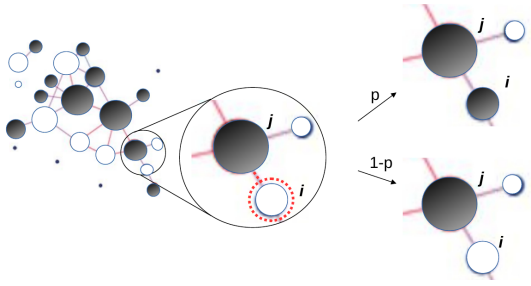

Figure 41: Illustration of the time-step of the HOpS model: firstly, active node i and its neighbor j are randomly chosen; secondly, a state of an active node, opinion $op_i(t)$, is changed with probability p depending on the status difference for nodes i and j. A node status is encoded by a node size.

Note that a probability *function of an opinion change* was chosen to be a sigmoid function $\tanh \sigma|st_j - st_i|$, since a sigmoid function represents the increasing likelihood of imitation processes to take place with an increase in the status difference [39, 251, 264].

It is important to explain a meaning of a novel *heterogeneity* parameter of the HOpS model. The reason for introducing a new system's "heterogeneity" parameter can be seen from the following observation. Let us consider a group with one strong leader-dictator with a very high status, where the information transmission is directed from the group leader to others, in contrary to a homogeneous group. By the same token, it has been noticed in [256] that a hierarchy in society induces information spread from the leader to others more efficiently, than in structures where the hierarchical structure is less "pronounced". Indeed, a tree-like hierarchical structure without loops Fig. 42 (a, b, c) admits smaller speed of convergence towards the equilibrium, than networks with loops, as in Fig. 42 (d) since there are less number of ways which random walk can take. Note, that it is also assumed that when all statuses are the same, an active node changes its opinion to an opinion of its random neighboring node with probability 0.5, and with equal probability opinion of an active nodes stays the same.

To summarize, the HOpS model is a particular kind of DN model with two prominent novel characteristics:

1. *Each node i has status st_i*, which is as a characteristic of a so-called social influence. A distribution of nodes' statuses introduces heterogeneity to the structure of a DN model and to a *mechanism of node state change*.
2. *Opinion change of each node i* is introduced by a threshold function and induces heterogeneity to a *mechanism of opinion spreading*.

In Section 13 I present a new *methodological framework for dynamical network models* revealing analytic solutions for the HOpS model on symmetric networks. As the next step, I consider the HOpS model dynamics on random Erdos-Renyi networks [77], Subsection 13.4.

13 Results for Heterogeneous Opinion Status (HOpS) model

Reality is what we take to be true.

David Bohm

The *analytical* solutions for the Heterogeneous Opinion Status model for particular networks topologies are introduced in Subsection 13.2. The *numerical* results for HOpS dynamics are described in Subsection 13.4.

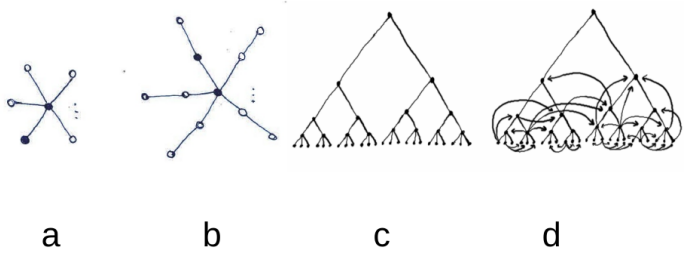

Figure 42: Star-like networks: simple star $k = 5$ (a), complex star (b). Hierarchical networks of different types: symmetric tree (c) tree with additional links between different hierarchal layers (d).

13.1 Analytic results for the HOpS model: dynamics on symmetric networks

...they were walking and walking...

"Moomins" Tuve Janson

There has been a variety of numerical studies on dynamics on networks, while analytic approaches to DN models analysis always have been lacking. Here I introduce a novel approach to study DN models using notations from theory of generalized cellular automata, Markov chains, and illustrating this approach on the HOpS model. *The main idea* of this technique is that for some model configurations, it is possible to calculate analytic solutions due to topological properties of these configurations. Let us call such configurations *basic configurations*. Which are these configurations? It is natural, first to consider basic network structures, particularly, a class of *symmetric networks*. Then further one can generalize model solutions for more complex underlying networks. The intuitive notion of a *graph symmetry* can be detected by graph measures [117] and is characterized by features of group of graph automorphisms [109]. For symmetric networks this group is non-trivial [106]. A formal definition for symmetric networks is as follows.

Definition. Two nodes u and v of a graph G are *similar*, if for some automorphism α of G, $\alpha(u) = v$. A fixed point is not similar to any other point. Two lines

$x_1 = u_1v_1$ and $x_2 = u_2v_2$ are called similar if there is an automorphism α of G such that $\alpha(\{u_1, v_1\}) = \{u_2, u_2\}$. Only graphs without isolated points are considered. A graph is *point-symmetric*, if every pair of points are similar; it is *line-symmetric* if every pair of lines are similar; and it is *symmetric* if it is both point-symmetric and line-symmetric [109].
Coming back to notations in Section 12, a *state of the HOpS model* at time step t is denoted as $G(V, Op(t), St, E)$ and is determined by set of nodes' states $Op(t)$. The values of opinion distribution $Op(t) = \{op_i(t), i \in [1, N]\}$ are components of a state vector at each time step. The state vector $Op(t)$ depends on a fixed statuses distribution St, a network topology $G(V, E)$, the initial opinions at 0 time step $\{op_i(0), i \in [1, N]\}$ and on the time-step characteristics. The function F describing a change of state-vector $F : G(V, Op(t), St, E) \to G(V, Op(t + 1), St, E)$ is contingent on the network topology.
It has been noticed that evolution of some processes on symmetric network topologies without loops has peculiar properties [136]. At the same time, topological properties of networks, such as symmetry, influence the main parameters, quantitatively characterize random walk on networks. These characteristics are, for instance, hitting time, cover time, mixing rate [157]. The classical theory of random walks deals with random walks on simple, but infinite graphs, like grids, and usually studies their qualitative behavior: does the random walk return to its starting point with probability one or if it returns infinitely often? Or how *structural or topological properties of networks* are related to properties of transformation matrices of random walks [28, 70]? An example of random walk properties is the mean quadratic derivation [133, 247], the characteristic time, i.e. time after which the random walk has passed through all the nodes, defined for finite networks [35].
With this in mind, first, I consider the HOpS model dynamics on symmetric networks without loops, for which I use the random walk theory [186, 195, 238] and demonstrate the HOpS model results, conducted using picture of discrete-time random walk, Subsections 13.2 and 13.3.

13.2 The HOpS model dynamics on linear networks

As a starting point, I reveal analytic solutions for the HOpS model for particular kinds of symmetric networks: linear and star-like networks.

Analytic solution for the HOpS model on linear networks

...who needs fish if you've got caviar?

J.Brodskij

Here I consider the HOpS model on linear networks, explained in two following propositions. Further term "model" is meant to be the HOpS model if not stated otherwise.

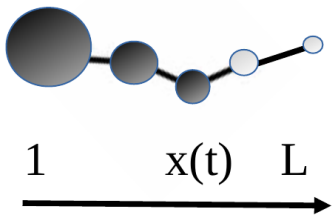

Figure 43: The HOpS model on a linear network for $L = 5$ nodes. A position of a border $x(t)$ between black and white nodes ($0 < x(t) < L$) is considered as a random walker. The special initial condition: node statuses decrease linearly, nodes with different opinions are separated by a border. The HOpS model, starting from such initial condition, reaches one of two final stable states: all nodes have the same color.

Let us consider the HOpS model on a *linear network* of length L. A space of all possible states of the HOpS model is denoted by $\mathbf{S}$, where each model state is fully described by opinion state vector is $Op(t) = (op_1(t), ..., op_N(t))$, where $op_i(t)$ is opinion of a node i. Starting from random initial conditions a phase space $\mathbf{S}$ has 2^L possible distinguishable states.

Definition. *A state of the DN model* at time step t is defined by a state vector of opinions $Op(t) = (op_1(t), ...op_N(t))$, if other model characteristics are fixed, such as status distribution $St = (st_1, ...st_N)$ or network topology.

Let us now assume, that the model starts from a *special initial model configuration*: $Op(0) = (op_1(0), ...op_N(0)) = (0, 0, ..., 0, 1, ...1)$ such that $x(t)$ left nodes are black, $L - x(t)$ right nodes are white. Moreover we assume, that nodes' statuses are linearly decreasing from black to white nodes, such that $\forall i \in [1, L-1]$ a status difference is fixed $|st_i - st_{i+1}| = \Delta_{st}$, Fig. 43.

Proposition I.

Starting from a *special initial model configuration*, all model states belong to a subspace S' of a phase space $\mathbf{S}$: $S' \subset \mathbf{S}$. Such a subspace is called an invariant subspace, since it fulfills the condition, that for any vector-state $Op(t) \in S' : Op(t+1) \in S'$ $\forall t$. The number of states in this subspace $|S'| = L$.

Proof: The number of states for the invariant subspace S' equals the number of all possible positions of the border $x(t)$ between black and white nodes. Starting from the special initial condition, the model is able to reach only a subspace of all system states, which belong to so-called invariant subspace. Hence, finding an invariant subspace of the system allows to describe all possible model states, or in other words, full phase space.

Proposition II.

The HOpS model dynamics with the special initial condition is equivalent to dynamics of an asymmetric bounded random walk $x(t)$ on a linear network.

Proof: Let us consider the probability of any black node to be converted into a white

node is equivalent to $(0.5 \tanh \sigma \Delta_{st} + 0.5)$, where $\Delta_{st} = |st_i - st_j|$ is a fixed status difference. Then a state of the whole system is described just by position of a random walker $x(t)$. A probability of a random walk to drift to the right is denoted by $a = 0.5 \tanh \sigma \Delta_{st} + 0.5$ and probability of a random walk drift to the left is denoted by $b = 1 - a$. The model has two equilibrium states: when all nodes are either all black or all white. A probability of a random walker to reach the right border is equal to a probability of the HOpS model to come to an equilibrium when all nodes are black.
Now the research questions on DN models behavior, Subsection 11.1, are started being translated to the language of the random walk theory.

Bounded asymmetric random walk on a linear network
As it has been previously shown, the HOpS model dynamics on a linear network with a special initial configuration, as in Fig. 43, is described by a *random walker* $x(t)$. The probability of a random walker $x(t)$ to be shifted to the right equals a, and the probability of a random walker $x(t)$ to be shifted to the left equals b (Proposition II). Then probability $p(x(t+1) = i)$ for an *asymmetric random walk* $x(t)$ to be in position i at time step $t + 1$ can be written as:

$$p(x(t+1) = i) = \begin{Bmatrix} 0 : |x(t+1) - x(t)| > 1 \\ a : x(t+1) - x(t) = 1 \\ b : x(t+1) - x(t) = -1 \\ 1 - a - b : x(t+1) - x(t) = 0 \end{Bmatrix} \tag{39}$$

For convenience let us set $a + b = 1$, which corresponds to a case when a random walker cannot stay on the same node. All together, this defines a transformation matrix P with size $|P| = |S'| \times |S'| = L \times L$. The non-zero entries of a matrix P are values on diagonals parallel to the main diagonal. Then an evolution equation for state vectors is defined by a tridiagonal right-stochastic matrix P:

$$Op(t+1) = Op(t) \cdot P \tag{40}$$

where $Op(t)$ is a state vector of opinions at time step t, and P is a transformation matrix (column-stochastic) of a corresponding Markov chain. Hence, estimating asymptotics of the HOpS model is equivalent to Gambler's ruin problem [263], which describes an asymmetric random walk on the integers $(1, ..., L)$, with absorption at 1 and L nodes. Solving the Gambler's ruin problem, we find solutions for the HOpS model on linear networks, as described below.

Proposition III.
Let us consider a bounded random walker on $[1, L]$ interval, starting from position x_0 with probability a to walk to the right and probability b to walk to the left. Then an asymptotic *solution for an asymmetric bounded random walk on a linear network* is

given by a probability to hit the right border:

$$p(x_0, a) = \frac{(a^{x_0}(1-a)^{L-x_0} - a^L)}{((1-a)^L - a^L)}. \tag{41}$$

Proof: Let a random walker be initially in position x_0. $p_i(j)$ defines a probability starting from i to hit j. It is easy to see, that $p_0(0) = 1, p_0(L) = 0$ and correspondingly $p_i(0) = ap_{i-1}(0) + bp_{i+1}(0)$. Then a characteristic equation is

$$ax^2 - x + b = (x-1)(ax-b)$$

which has roots $\{1, b/a\}$. For $a = b$ a random walker becomes symmetric. For $a \neq b$ a general solution $p(x_0, a)$ is sum of the roots with the coefficients defined by the absorbing states at 1 and L. Thus the probability of a random walker to hit one of the borders is:

$$p(x_0, a) = \frac{(a^{x_0}(1-a)^{L-x_0} - a^L)}{((1-a)^L - a^L)}. \tag{42}$$

Moreover the Gambler's ruin problem can be viewed as a special case of a first passage time problem, which asks to compute the probability that a Markov chain, initially in state, hits one fixed state before another.

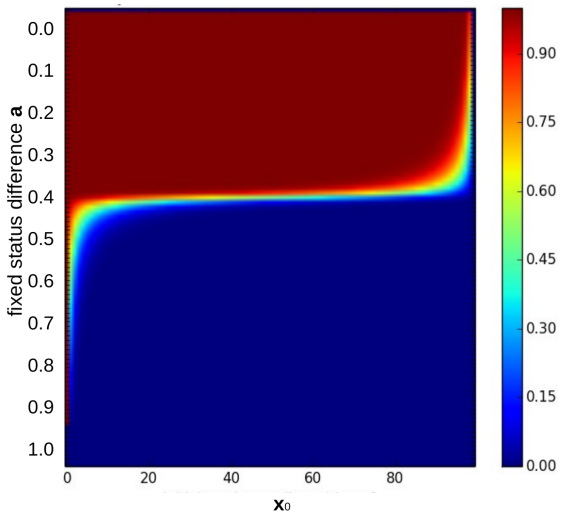

Figure 44: Dynamics of the HOpS model on a linear network. $x_0 \in [0, 100]$ (horisontal axis) is initial number of black nodes and $a = 0.5 \tanh \sigma \Delta_{st} + 0.5$ (vertical axis). Colorbar corresponds to the probability $p(x_0, a)$ for the system to come to one certain equilibrium, when all nodes are black, starting from x_0.

To sum up, the analytical results for the HOpS model on a linear network are:
(i) *The HOpS model dynamics on a linear network is described by Propositions I,*

II, III. The quantitative characteristics of a phase space of the model are given in Proposition I. The formula (42) estimates the probability to reach a stable states of the model. From the formula (42) it is clear that a (a function $0.5 \tanh \sigma \Delta_{st} + 0.5$) characterizes a speed of the model convergence towards an equilibrium and σ denotes scaling of a spreading process on a network.
(ii) *The analytic result of Proposition III is illustrated by the numerical result, Fig. 44.* Each model simulation is made for values of x_0 and a. $x_0 \in [0, 100]$ corresponds to initial number of black nodes. $a = 0.5 \tanh \sigma \Delta_{st} + 0.5 \in [0, 1]$ characterizes the status difference between nodes. Then a probability to find the model in one certain stable state numerically corresponds to a ratio between a number of model simulations, which reach one certain possible equilibrium state to a number of total model simulations. The probability to find the model in its final state is marked by the color of each point (x_0, a), Fig. 44. Red region above the yellow curve on Fig. 44 corresponds to model simulations when the model converges towards one equilibrium, or in other words, a random walker reaches the right border. The curve separating red and blue regions is implicitly defined via relation $p(x_0, a) = 0.5$, Eq.(42), which gives the formula for the curve: $a^{x_0}(1-a)^{L-x_0} = 2(1-a)^L - a^L$. Blue region below the curve corresponds to another absorbing state when the model reaches another stable state and, hence, a random walker with the characteristics from that region never reaches the right border.
(iii) *The schematic diagram of a discrete phase space of the HOpS model on linear networks is presented in Fig. 45.* The arrows on the diagram correspond to transitions between different model states. Topology of a diagram of the model on a linear underlying network is trivial, yet it illustrates how one can represent a part of a phase space of DN models. For more convoluted underlying network topologies the model phase space has more complex structure, as it is shown in Subsection 13.3.

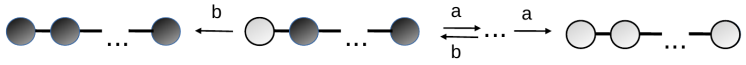

Figure 45: Schematic diagram of a part of a full phase space of the HOpS model on a linear network is presented as a sequence of states and transitions between them. Here the phase space is shown only for the HOpS model with a special initial state: $L - x(0)$ nodes from the left border are black, and $x(0)$ nodes from the right border are white at time step $t = 0$.

Additionally to the analytical results, spectra of transformation matrices P for various values of parameter $a = 0.5 \tanh \sigma \Delta_{st} + 0.5$ are calculated in Fig. 46. Interestingly, spectral properties of a transformation matrix and mixing properties of the system, described by this transformation matrix, are related. The *spectral gap*, by definition, is a gap between the largest and the second largest eigenvalues of a matrix. As can be observed from Fig. 46, the spectral gap is smaller for larger a values ($a > 0.5$), which means that larger values of parameter a correspond faster mixing times of the system [49] and forces faster reaching the equilibrium than for

smaller a values ($a < 0.5$). Translating this to the language of the HOpS model, the bigger the status difference Δ_{st}, the faster the equilibrium state is reached. This property is also related to the mixing time of the corresponding Markov chain and it is also known as Cheeger Inequality [157].

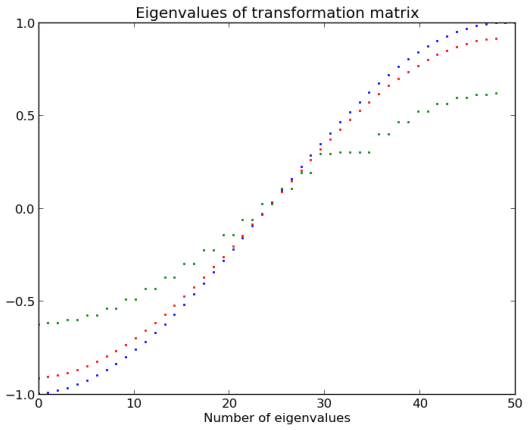

Figure 46: Spectrum of a transformation matrix for the HOpS model on a linear network for $L = 50$. Larger spectral gaps are observed for larger parameter values $a = 0.5 \tanh \sigma\Delta_{st} + 0.5$ values, as the result, this gives faster convergence towards the equilibrium. Spectra for $a = 0.5$ is blue dashed line, for $a = 0.7$ - red dashed line, for $a = 0.9$ - green dashed line.

13.3 The HOpS model on star-like networks

After demonstrating analytical solutions for the HOpS model dynamics on a linear network, the next step is to consider the HOpS model on more general symmetric structures, such as star-like networks, Fig. 42 (a,b). First I consider particular types of *star-like networks*.

Definition. A *simple star* is a network with one central node and k "leaves" i.e. one-node edges, attached to a central node, Fig. 42 (a). A simple star is a tree-like network with tree depth 1.

It is important to emphasize, that the definition of a star-like network highlights two main differences in comparison with a linear network: (1) We have to cope with a more complex network topology. Moreover dynamics on it is not equivalent to a random walk dynamics in the case of linear network. (2) As the consequence, the over-all complexity of the model dynamics on star-like network is larger. However, as it is found below, the analytic techniques to describe the model dynamics on simple star-like networks are originated from the framework for the linear network case, Subsection 13.2.

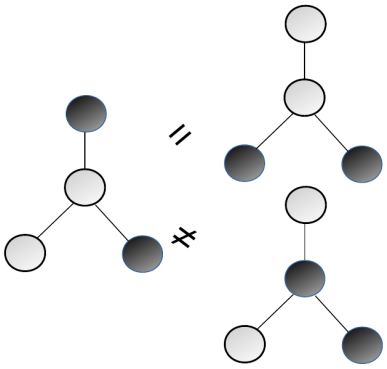

Figure 47: Equal states of the HOpS model on a *simple star* with $k = 3$ leaves: the states are equivalent iff in both states the central node has the same color and the number of nodes-leaves with identical color is the same.

The model dynamics on a simple star network

Let us first consider a *simple star network* with k one-node "leaves" and one center. The number of the system's states for random initial conditions is $|\mathbf{S}| = 2^{k+1}$. Now assume, that all "leaves" of a simple star network have some fixed statuses $st_i = s$, $i = \{1, ...k\}$, and a central node has a higher status $st_{k+1} = s + \Delta s$ (Δs is set as a parameter of status difference). As we saw for the model on a linear network, the existence of the invariant subspace simplifies the description of the phase space $\mathbf{S}$ of the system. Remarkably, there is no non-trivial invariant subspace inside space $\mathbf{S}$, for simple star-like networks. If there would exist such an invariant subspace S', then there would exist a special initial condition, i.e. a vector-state $Op(t)$ from such a subspace $Op(t) \in S'$ such that for any transformation P: $Op(t)P \in S'$ $\forall t$, but by finite enumeration method it is easy to see that there is no such initial conditions, in contrary to the case for the HOpS model on a linear network. Simply speaking, the reason for this is that a structure of group of symmetries of a star-like network is more complex. Nevertheless, it is possible to "simplify" a space $\mathbf{S}$ using a natural algebraic technique to induce the parametrization on a space $\mathbf{S}$ [258]. The main advantage of the parametrization is that it allows to change the structure of the space $\mathbf{S}$, so that the parametrized space S^* has an invariant subspace, while the corresponding "initial" space $\mathbf{S}$ doesn't. Let us first consider elements of the space $\mathbf{S}$, the states of the model at time steps t_i and t_j with corresponding state vectors $Op(t_i)$ and $Op(t_j)$. In order to parameterize a full space of states $\mathbf{S}$ I introduce a natural equivalence relation between states.

Definition. I call two states of the HOpS model on the star-like network *equivalent*, as in Fig. 47, $Op(t_i) \sim Op(t_j)$ iff:

in both states the central node has the same color;

in both states the number of leaves-nodes with white color is the same.

Later on I come back to this, discussing equivalence of DN models. Thus using such equivalence relation, we are now ready to parametrize the space of states $\mathbf{S}$.

Parameterized space S^* is then defined as $S^* = \mathbf{S}/\sim$. Notably, a group of equal states of the space $\mathbf{S}$ corresponds to one state of the space S^*. Let us call states of space S^* *macro-states*, in order to distinguish them from states of "initial" phase space $\mathbf{S}$.
In the following proposition I describe equivalent macrostates from the algebraic point of view.
Proposition IV.
All equivalent macro-states of the model on star-like network form a group Π' in respect to the operation of a permutation.
Proof. Let us consider equivalent states $Op(t_i)$ and $Op(t_j)$ from the whole discrete phase space $\mathbf{S}$ of the model on a star-network. Each node has a Boolean value $op_i \in \{0, 1\}$. Then the proposition follows from the fact that vector states $Op(t_j)$ and $Op(t_j)$ belong the same macro-state in the parametrized space S^*. In another words, a permutation of nodes' states generates a state, which belongs the same macro-state iff:

$$\exists \pi \in \Pi' : (op_1(t_i), ...op_{k+1}(t_i)) = (\pi(op_1(t_j)), ..., \pi(op_{k+1}(t_j))). \tag{43}$$

Without loss of generality, let us enumerate the central node as the 1^{st} node with opinion, denoted by $op_1(t)$. Then a permutation on states of nodes $\pi \in \Pi'$, which transforms two equal states between each other, has a property:
$\pi(op_1(t_i), op_2(t_i), ...op_{k+1}(t_i)) = (op_1(t_i), \pi(op_2(t_i)), ..., \pi(op_{k+1}(t_i)))$, so that opinion of the central node stays the same $op_1(t_i) = op_1(t_k)$ for any t_i. In other words, the group Π' consists of sequences of length $k + 1$ $\{0, 1\}$, for which a value of the central node is preserved.
Interestingly, a subgroup Π' of some bigger group Π is isomorphic to a subgroup of symmetric group. This follows from the Caley theorem [14], which states that every finite group Φ is isomorphic to a subgroup of a symmetric group $Sym(\Phi)$. This property of a group of permutations gives intuition behind structure of permutations.
Furthermore, a transformation of a state vector $Op(t)$ can be represented as
$(op_1(t), ...op_{k+1}(t)) \to (op'_1(t + 1), ..., op'_{k+1}(t + 1))$, where an opinion of a randomly chosen *active node* i can change to an opinion of an opposite node: $op_i(t + 1) = (op_i(t) + 1) mod 2$.
The discrete phase space of the model on a star-like network is shown in a schematic way in Fig. 48, where each model configuration can be transported to another model configuration with probabilities a or b correspondingly. In fact, the structure of such graphical diagram is not occasional, and is connected to algebra of processes [40] and sequential dynamical networks [20].

Sequential dynamical systems

Definition. *Sequential dynamical systems (SDSs)* are constructed from the following components: 1. A finite underlying graph G with vertex set $V = \{1, 2, ..., N\}$. Depending on the context the graph can be directed or undirected. 2. A state x_w

for each vertex i of G taken from a finite set of values K. The system state is the N-tuple $x = (x_1, x_2, ..., x_N)$, and $x[i]$ is the tuple consisting of the states associated to the vertices in the 1-neighborhood of i in G (in some fixed order). 3. A vertex function f_i for each vertex i. The vertex function maps the state of vertex i at time t to the vertex state at time $t+1$ based on the states associated to the 1-neighborhood of i in G. *Stochastic Sequential dynamical system (SSDS)* has the same components as SDS accept that the update rule has stochastic component, according to [160].
Sequential dynamical systems may be thought of as generalized cellular automata, the main difference between SDS and the DN model is that SDS has a deterministic update rule. Similarly to the case of the HOpS model a structure of a phase space of SDS is governed by topological properties of an underlying graph G, vertices' states $\{f_i\}_{i\in[1,N]}$, and the so-called update sequence ω defining the transformation of vertices' states. Note that for deterministic SDSs, each state in its phase space can be transformed only to one state, while for stochastic SDSs (SSDS) each state does not necessarily have only one possible state, where it can be transformed. In the definition of DN model, a transformation F is defined on a set of all possible states $\Gamma = \{G(V, Op(t), St, E), t \in [0, \infty)\}$ so that $F : \Gamma \to \Gamma$. Note, that a transformation matrix of a function F is denoted by P.Another useful notion from theory of SDSs is a *digraph of SDS*, a graph, where each link of digraph is associated with a transition between model states in discrete phase space of SDS. In the following proposition I explain the connection between SDSs and the HOpS model.
Proposition V.
A phase space of the HOpS model is associated with a digraph of some stochastic SDS.
Proof of this short proposition follows from the definition of a *stochastic SDS*.

A phase space of stochastic SDS can be understood as a weighted underlying graph, where weights denote a probability of transformations between states of the system.
Definition. The *basin of attraction* of an attractor in a discrete phase space of SDS is the set of all states that eventually end up on this attractor, including the attractor states themselves. The size of the basin of attraction is the number of states belonging to it.
In case of the HOpS model on a finite networks a basin of attractor consists of all states, which can be transformed to absorbing states (consensus states). The HOpS model on a linear network has two possible absorbing states, which depend on the initial condition.
Definition. *The state space* Ω of the HOpS model (or of any finite DN model) is the finite directed graph (digraph), where an edge exists between states, if they can be transformed from one to another.
The following proposition describes the HOpS model from the point of view of discrete finite dynamical systems [182].
Proposition VI.
Since the set of states of the HOpS model (in principle, of any finite DN model) on a finite underlying network is finite, any directed path must eventually enter a directed

cycle, called a limit cycle.
Proof: Directed paths in full space of the HOpS states Γ correspond to iterations of a function F on the model's states or to state, at the beginning of the path. Then because there is a finite number of possibilities of paths, at some point path of states comes back to the state, where it started. In other words, because we deal with the map between the set of all states $\{0,1\}^N \to \{0,1\}^N$ and the sets of possible states are finite, for any given initial condition it must have either a unique fixed-point or a limit cycle, otherwise the set would have to be non-finite.

Definition. Two SDSs are called *isomorphic* if there exists a digraph isomorphism between their phase spaces. These SDSs are stably isomorphic if there exists a digraph isomorphism between their limit cycle graphs.
Finally, let us come to the HOpS dynamics. Let us call two DN network models *equivalent in terms of the HOpS model*, if their digraphs of phase spaces of these models are isomorphic. Then Propositions I-VI can be extended to describe the HOpS model dynamics on a broader class of underlying networks, than just on linear and star-like networks, but also on the underlying networks, for which the HOpS model phase space is equivalent to a phase space on star-like network or linear network. Topological properties of the digraph, as in Fig. 48, highlight symmetric structure of the HOpS model phase space, which, in fact, is connected to symmetric properties of underlying networks. In particular, the diagram in Fig. 48 exposes two absorbing states, transient states and moreover contains a limit cycle. Hence, SDSs theory provides a promising tool to describe DN models. Further I come back to analysis of the HOpS model in terms of transformation matrices.

Transformation matrix approach for the HOpS model on star-like networks

Let us now consider a transformation matrix for the model macro-states denoted by P_k, where k is the number of leaves of the simple star-like network. The size of matrix P_k is defined by the parametrized space S^*, as I showed in the case of the model on linear networks. Therefore a size of the transformation matrix $|P_k|$ for a star-like network is $(2k+2) \times (2k+2)$, where $2k+2$ is the total number of states for the model on a star-network with k leaves. This can be easily seen from the diagram Fig. 48. As an example, I consider the *HOpS model on a simple star network* for $k = 3$ leaves. which has a transformation matrix $|P_3| = 8 \times 8$. The structure of P_3 can be reordered in such a way that the first and the last matrix rows correspond to absorbing states of the system and rows $i \in [2, 2k+1]$ of the matrix P_k correspond to transient states. The elements of P_3 are rearranged in such a way that the nonzero

matrix elements a, b are parallel to the main diagonal. Then the matrix has a form:

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & b \\ a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

In general, if the simple star-like network has k leaves, the transformation matrix P_k has Toeplitz structure. This Toeplitz matrix has nonzero elements on four diagonals parallel to the main diagonal, which correspond to transition probabilities a, b between states of the model:

$$P_k = \begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 \\ a & 0 & 0 & 0 & ... & b & 0 & ... & 0 & 0 \\ 0 & a & 0 & 0 & ... & 0 & b & ... & 0 & 0 \\ & & & & ... & & & & & \\ a & 0 & 0 & 0 & ... & b & 0 & ... & 0 & 0 \\ 0 & a & 0 & 0 & ... & 0 & b & ... & 0 & 0 \\ & & & & ... & & & & & \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 1 \end{pmatrix} \tag{45}$$

where the distance between nonzero elements a and b elements in each row equals $k + 1$. In analogy with the case of the HOpS model on a linear network knowing the transformation matrix P_k of the system allows to find a stationary solution. It can be found as a vector state $Op_{t\to\infty} = Op(t)$ for $t \to \infty$ for the HOpS model on a star-like network by the evolution equation (40): $Op(t + 1) = Op(t)P_k$ and $Op_{t\to\infty} = Op(0)(I - P_k)^{-1}$, where $Op(0)$ is an initial state of the parameterized space S^*, I is an identity matrix.

Symmetric networks have nontrivial group of automorphisms (consisting more than from one element by the definition). This fact also causes that the invariant subspace of the full space of states is nontrivial, therefore one obtains some symmetry in the structure of the phase space and it is possible to describe network state using schematic diagrams, such as in Fig. 48. Although in this section I considered *linear and star-like networks*, some analytical results from Subsection 13.1 are generalizable for solutions on the HOpS model dynamics on more general topologies. One example of such topology is a *complex star* network with k "long leaves", chains formed by l-nodes attached to the central node, Fig. 42 (b). Ultimately, presented methodological framework gives possibility to perform analysis of other DN models for network's topologies such as linear combination of star-networks or trees, circular networks, fully connected graphs and others.

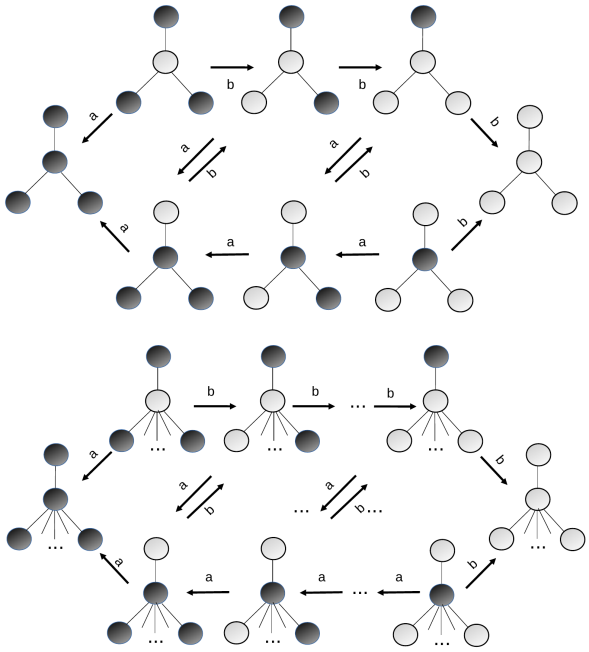

Figure 48: Phase space of the HOpS model on a *simple star* with $k = 3$ leaves (top), arbitrary number of leaves (bottom). The arrows between each two states of the HOpS model represent transition probabilities, which equal a and b correspondingly, Subsection 13.1.

To summarize the main analytical findings of this section are:
(i) *The HOpS model dynamics on star-like networks is described* by Propositions IV and V, the discrete phase space of the model is demonstrated in the schematic diagram, Fig. 48. As it can be seen from Fig. 48, some states of the HOpS model form the infinite cycles. These states are transient states of the HOpS model on a star-network, where not all nodes-leaves have the same color.
(ii) *The evolution of the HOpS model can be described by the transformation matrix equations.* An example is a transformation matrix for the HOpS model on a star network, Eq.(44).
(iii) *The general framework for studying of a dynamical model on networks* can be formulated as the following. Firstly, one needs to find possible symmetries of the underlying network itself and hence to find the invariant subspace of the full space of model states, as it has been demonstrated for a particular cases of networks in Proposition I. Then a phase space (or a part of a phase space) of a model can be represented as a digraph of SDSs. as it was described in Propositions V and VI. This would give qualitative characteristic of a model evolution. Secondly, one need to find the transformation matrix between all states, which would give quantitative characteristic of a model evolution.

13.4 Numerical results for the HOpS model on random networks

The analytic solutions for DN models on special networks topologies were discussed in details in Section 13. Further step is to get an intuition for the *numerical results* of the HOpS model on random network topologies. Apart from inherent topological properties of random networks, these networks usually have a self-induced structure, which influences the flow of transport [120]. This gives additional motivation to investigate such network models in the context of the HOpS model.

The HOpS model dynamics on random Erdos-Renyi networks

The random network type of the interest is random *Erdos-Renyi (ER) network* $G_{N,p}$ on N nodes [77]. As it was already defined in Chapter II, each possible edge between two vertices is present in ER network with independent probability p, and absent with probability $1-p$. It is important to mention that each realisation of ER network with particular p value is only a particular member of the entire statistical ensemble of random ER graphs. Hence, studying such model dynamics one needs to consider statistical ensemble of random networks instead of one particular random network realisation. Up till now in this chapter the space of model states of the HOpS model was mostly analyzed analytically. Underlying structure of ER networks is more complex than in "deterministic" symmetric networks, which leads to more intricate organisation of a phase space. I numerically examine behavior of the HOpS model by changing the model control parameters, specifically, the ER network randomization parameter p. Then for each specific value of a control parameter p I calculate the number of time steps till consensus is reached, so-called *waiting* or *relaxation time*. Notably, relaxation times for the HOpS model on symmetric deterministic networks with analytically estimated transformation matrix P can be estimated as a spectral gap of a matrix P, while waiting time for ER networks here is estimated numerically.

The waiting time for the HOpS model on ER networks.
The waiting time as the function of an ER parameter p for each random configuration of network is shown in Fig. 49. The HOpS model dynamics is simulated on each realisation of 300 ER networks $G(N,p)$ for $N = 90$ nodes for each value p. The initial opinion distribution is randomized at each model run. doesn't significantly depend on a number of nodes. The numerical simulations are performed until the HOpS model reaches an equilibrium. The waiting time in Fig. 49 reaches the plateau for a value $p \approx 0.15$. But first let us discuss some properties of ER networks.
Interestingly, ER networks have rich geometric properties, while being constructed by quite a simple rule. Let us consider the process of slow increase of p. First, for low p values ER network consists from disconnected small components of $G_{N,p}$, in each of them the equilibrium in average is reached faster than on the whole fully connected network $G_{N,p}$, because these subcomponents are smaller [140]. Then if p increases, the giant component will be formed. For larger p all disconnected components will form a joined component, which still has a low link density, therefore the waiting

time for the HOpS model is expected to increase. When p continues to increase, it forms a fully connected component, and because of its high connectivity the waiting time is decreasing. All in all, for $p < p_c = 1/N$, clusters are small and tree-like. For $p > p_c$, there exists a "giant" cluster that consists of a non-zero fraction of all nodes [191]. Using kinetic approach [140] one gets that the cluster size distribution c_k is given by $c_k(t) = \frac{k^{k-2}}{k!} t^{k-1} e^{-kt}$, where "relative" time defines the rate at which links have been introduced to ER network. One may look at the percolation phenomenon in ER networks from the view of geometric properties: we observe that N "seeds", i.e. nodes of ER network $G(N, p)$, which form disconnected clusters, emerge into a "forest", i.e. a complex giant component, which is formed from cycles and trees linked together. As the result of such reformation of clusters size distribution, the waiting time also is affected by it, which is discussed in the following remark.

Remark. The waiting time for the HOpS model dynamics on ER networks exhibits a transition in respect to the value of the ER parameter p. As it has been found numerically in Fig. 49, the transition of the waiting time is observed for value $p \approx 0.15$ (this value is robust against number of nodes N). Some intuition for the analytical explanation to this observation is the following. Let us consider an ordinary diffusion process on a densely connected network [94]. Let T_d denote the waiting time till complete spreading through all nodes. Now let us compare T_d with the waiting time T_c, which is the time of a complete spreading for a less densely connected network (which still forms a joined component) on the same set of nodes. Intuitively it is clear that the density of network connections influences the characteristics of dynamic process on networks, giving $T_d < T_c$, if other parameters, such as number of nodes in a network or network connectivity, are kept the same for both networks. In general, dynamics of the HOpS model on arbitrary underlying structures is a challenging question, which is discussed further in the conclusions and the outlook.

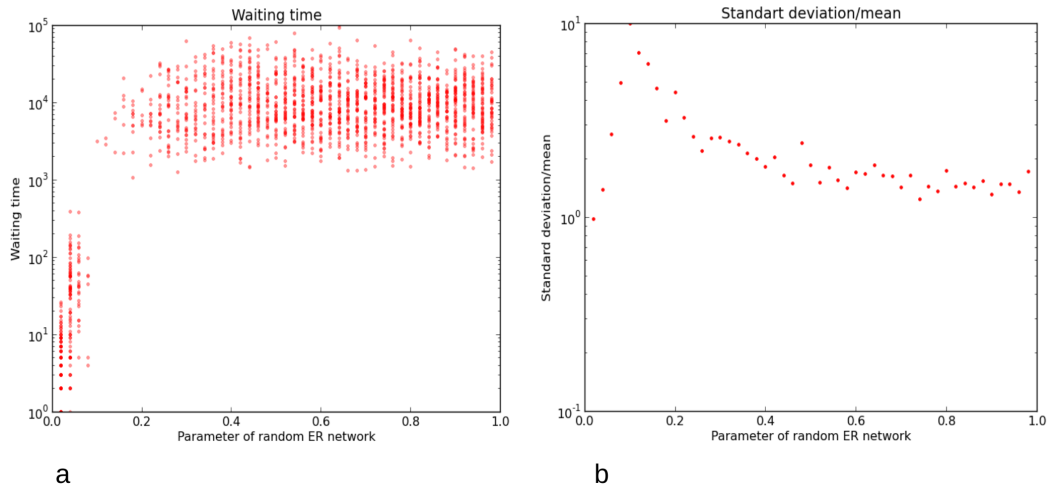

Figure 49: ***The waiting time*** **for the HOpS model on random Erdos-Renyi networks depends on the probability of link creation** p, subplot (a). Random realisations of the model are made for fixed parameters: $N = 90$ nodes, $T_{max} = 100000$, $\sigma = 1$, with random statuses distribution for each realisation. The waiting time (vertical axis) is calculated for each parameter p (horisontal axis). *Log-log* plot for the standard deviation/mean values of the waiting time, subplot (b), where a peak for parameter of random ER network $p \approx 0.15$ indicates the transition.

14 Conclusions and further directions

> The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom. But how much of the nose on your face can you see, unless someone holds a mirror up to you?
>
> Aizek Azimov

The main conclusions of this chapter are enumerated below:

1. *A novel discrete state Heterogeneous OPinion-Status Model (HOpS) was introduced.* The model dynamics on specific network topologies is described by Propositions I-VI. A phase space of the HOpS model is analyzed using theory of random walks and can be extended using theory of sequential dynamical systems.

2. *The HOpS model serves as an revealing test case for the new theoretical framework to describe a phase space of a discrete state model on networks*, as described in Section 13. This general setting links problems of DN models with theory random walks. For symmetric networks a phase space of the model is regular, Figs. 45 and 48. Moreover, in this chapter it has been found that the speed of convergence towards the equilibrium state for the dynamical

network model is affected by the heterogeneity parameter (here heterogeneity is introduced by a status distribution of nodes). In particular, the speed of convergence for the HOpS model on linear and star-like networks depends on the status difference parameter Δ_{st}, as it has been shown in Fig. 46.

3. *Here it has been found that a theory of SDSs suggests a possible approach to study DN model's dynamics.* This can be significant both theoretically and practically for a case of more general underlying graph structures. The novel approach to study DNs has some drawbacks, for instance, growing entanglement of a phase space with increasing complexity of a topological structure of the underlying network.

4. *The numerical results* for the model on Erdos-Renyi (ER) random graphs, are given in Subsection 13.4. The transition for a waiting time is observed for ER networks for an approximate value of ER randomization parameter $p \approx 0.15$, Fig. 49.

5. *The effects of heterogeneous spread of opinion on a network in the HOpS model were studied.* In the common words, the network topology is reflected in the HOpS model dynamics. A key notion of most of DN models is that simple behavioral local rules generate complex global model dynamics. A novel heterogeneity parameter of the HOpS model can be introduced in any kind of DN model, induces a novel complex model dynamics, as it has been illustrated for linear and star-like networks.
   All in all, most of the real world systems can be represented as a complex dynamical network model, therefore development of general techniques to study dynamical network models is a topical issue.

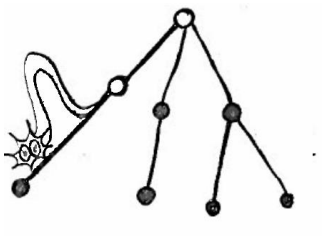

Figure 50: Change of nodes states in the HOpS model on a tree-like network can be presented as "random walkers", traveling on the network: at each time step one node has a probability to change its state.

Outlook

> Sing, goddess, the anger of Peleusâ son
> Achilleus and its devastation, which put pains
> thousandfold upon the Achaians, hurled in
> their multitudes to the house of Hades strong
> souls of heroes...
>
> "$I\lambda\lambda\iota\alpha\delta$", Homer

The outlook includes open questions and generalisations to this chapter of thesis:
1. *What is the general method for finding analytical solutions for the HOpS model on networks with arbitrary topology?* A novel technique to analyze DN models, presented here, has one particular limitation - the requirement of a regular or acyclic structure of an underlying network, like a tree-like structure on Fig. 50. However, obtained analytical solutions for the model can be extended in a quite straightforward way further even for networks with cycles. Also, in order to find the HOpS model solutions on asymmetric networks, adjacency matrices of asymmetric networks can be tackled as perturbed adjacency matrices of symmetric networks. Moreover, it seems potentially interesting, to relate schematic diagrams, Fig. 48, with theory of group-reduced distributions [106] and the algebraic Galois theory for SDSs [149]. Additionally theory of Coxeter and Sylow's groups could serve as promising instrument for theoretical description of SDSs [159, 181], hence also for analytical description of DN models.

2. *Which are further directions for the HOpS model?* One potential generalisation of the model is to introduce the co-evolution of network topology and nodes' states. One of the possible model modification: let us call node i an active node, if the random walk on the network is traveling on the network through this node. Additionally to this, some problems from the random walks theory [134, 186, 234] can be linked to various problems on finding analytical solutions DN dynamics. For instance, random walks on weighted directed networks are closely interrelated, random walks on circular networks with randomly added short-cuts.

3. *How the HOpS and voter models are interrelated?* From the schematic diagram of the phase space of the model states of the HOpS model, Fig.48, it can be seen that in the HOpS model a node can change color from white to black even even if most of the nodes in its surrounding are white (this is not possible in the voter model, where the mean opinion of the surrounding nodes defines the change of the opinion). These models can be described using general framework [126]. What are the main differences in these models?

4. *Which are possible applications of kinetic approach for the HOpS model?* There is a link between the HOpS model on a linear network from random initial conditions and the domain walls dynamics. The analytic solutions for the domain walls dynamics are known as a sum of Bessel functions [140]. Let us now call a *domain wall* an edge in the HOpS model, which separates nodes with white and black nodes in the HOpS

model, or more generally and edge, which separated nodes with different states. For instance, in Fig. 44 there is one domain wall on a linear network, since there is only one edge between two sets of nodes with different opinions. In terms of the language of domain walls it is clear that the HOpS model on a linear network, which has an initial condition where there are several domain walls, has various dynamics rather than the HOpS model with one domain wall in the initial condition. The homogeneous case, when the domain walls can move symmetrically with equal probabilities on a linear network, was analytically considered in [138]. All in all, these processes are connected to so-called "quenched disorder" phenomenon.
Furthermore, the methods, developed in Chapters II, III for evolving networks and flow-networks, can be applied in order to quantify changes in evolving DN models on coevolving networks.

Conclusions

15 Summary of all chapters

> How two ex-students turned on to pure mathematics and found total happiness
>
> Knuth

In the thesis I developed several analytical techniques to analyze complex systems. In particular, Chapter II is dedicated to the development of new indices to analyze evolving networks. In Chapter III I introduced the theoretical method of correlation matrix analysis for flow systems, in Chapter IV I presented a new heterogeneous dynamical network model and demonstrated techniques to analyze dynamics on networks on particular network topologies.
Here I summarize and draw my conclusions from the results from all the previous chapters, and point out some open questions.

15.1 Evolving networks: methods of analysis for random models and data structures

> Can we predict the future, as birds predict the sunrise with their singing?
>
> Folklore

In Chapter II I introduced a novel *common component analysis* to characterize *evolving networks*. These new measures detect spatial variability patterns in evolving networks, which represent an evolving complex system. Developed network indices, i.e. CCEF function, and evolving network measures have been applied to various types of networks: ensembles of random and correlation networks, obtained from the temperature field of Asian Monsoon [212, 260] and two-dimensional dynamical systems [179, 253]. Examining the influence of spatial sampling for different node distributions it was found, that inhomogeneous spatial sampling can lead to distorted or misleading patterns in the network measures [179]. CC analysis was successfully tested on random networks and conceptual models [77, 213]
It is important to emphasize that using CC analysis I identified a high degree of spatio-temporal persistence in the year-long daily temperature anomaly correlation networks of the Asian Monsoon domain. Studies of the persistence of variability patterns with CC analysis coincide well with the previous studies on climate system

[24, 63, 100]. All in all, common component framework for functional networks evolving in time is a promising and useful tool for analysis of spatial and temporal transitions in climate and evolving complex structures.

15.2 Theoretical analysis of correlation networks

Vopreki techeniy reki

A.F.

In Chapter III I introduced a novel discrete *flow-network approach to analyze dynamical systems with underlying time-dependent velocity fields.* Firstly, I established the method to construct networks from the advection-diffusion equation (ADE) for temporal flows [254], overcoming restrictions of the previous approaches [147, 177], such as high computational complexity and limitation to stationary flows in the underlying system. In particular, the method of flow-networks is introduced for general systems driven by time-dependent velocity fields with Gaussian noise, temperature-decay and external forcing. Importantly, *the analytic formula for the correlation matrix was obtained* for the time-dependent underlying system. The flow-networks method was applied to meandering moving flows, which resemble the simplified velocity structure present in ocean currents. This allows to identify various flow regimes and characterize transport in the fluids such as various mixing regimes [187, 255]. The results of the analysis of correlation networks structure showed that correlation networks are not sensitive to steady sources and sinks and the profound impact of the signal decay rate on the network topology. This allows to give a full description to the information about process using correlation networks analysis.
All in all, a new *flow-networks technique* allows to draw a connection between network topology and properties of the underlying system [147, 254] using combination of the network measures, such as degree and clustering coefficients. Proposed framework of flow-networks, which are analytically derived from the underlying velocity field, provide a tool to understand the nature of relationship between system's dynamics, network's topology and network measures. Hence, it it relates the dynamical systems theory and graph theory. There is number of further possible applications of the flow-networks methods [215, 254] for ocean currents [38] and other systems, where particles transport plays a crucial role [25, 152].

15.3 Dynamics on networks

We were arguing energetically about whether the world is actually evolving...

Lancszemek, Karinthy Frigyes

In Chapter IV the new dynamical *heterogeneous opinion-status model (HOpS)* on a network has been introduced. Each node of this model has a certain quantifiable status, which influences the opinion dynamics taking place on a network. Despite the

relative simplicity of setup of the HOpS model or rather precisely because of it - this model produces rich dynamics and serves as an accessible test case for the methods for analysis of processes on networks. The so-called *discrete state models*, such as the HOpS model, represent diffusion of opinions and spread of diseases in society [43, 102]. I examined dynamics of diffusion-like processes on various types of networks and obtained analytical solutions for the HOpS model for precise examples of network topologies. In particular, in Chapter IV I have developed a method to study the dynamics on networks with heterogeneity parameter. Proposed framework allows to find analytical solutions of the model for series of symmetric acyclic graphs. The methodology relies on the theory of stochastic processes [134, 186, 201] and discrete dynamical systems and allows to learn more about the dynamics on graphs even with more general topologies [109, 133]. All in all, the introduced analytical technique to tackle discrete dynamics on networks consists of two parts: firstly, to find network topologies for which the model on network has simpler or, in other words, reduced dynamics (i.e. on regular networks), secondly, to construct the discrete phase space for the underlying network (the second step for regular symmetric networks without loops can be made analytically). I demonstrated this new technique, applying it to the HOpS model, determined by the control parameters: heterogeneous distributions of initial opinions, fixed statuses and network topology. Importantly, underlying network topology determines structure of the discrete phase space of the model, in particular, for symmetric underlying network without loops the phase space of states of the HOpS model can be drawn as a symmetric graph. Ultimately, I have found special types of underlying networks' topologies, for which the HOpS model dynamics can be analytically described. Finally, I translated problems for discrete state models to the language of sequential dynamical systems [149], which suggests new analytical description of processes of imitation and adaptation dynamics on complex network structures [97, 264].

16 Outlook

Carpe diem

Quintus Horatius Flaccus

There are several reasons for the acceleration of the interest to the graph theory and critical, collective effects on networks [109, 169, 222]. Nowadays, the research on complex systems is constantly progressing and is converging with different areas, such as network control [183], evolution of social connections [103, 232] and many others. Evolving networks remain an attractive topic for network analysis [66, 121, 253], for example, the generalization of static network measures to dynamic ones is an actual problem.

Developed techniques in Chapters II, III and IV can be combined with other analytic techniques, such as the ones, presented in [11, 99, 268]. Methods introduced in the thesis have their advantages and also drawbacks. First, the discrete flow-networks

method [254] is attached to a specific grid, however, thanks to its low computational cost, it allows high resolution calculations, which is a benefit from computational point of view in comparison to the previous existing methods [177, 179]. Second, the new HOpS model describes a particular kind of processes on networks, while the whole variety processes is obviously much broader. Presented methods can be generalized in order to overcome the limitations, listed above. In addition to several open problems, discussed at the end of each chapter, the suggested temporal network measures can be applied to ensembles of functional networks, which represent spatio-temporal evolution of systems, such as analytically derived flow-networks [147, 177, 229, 254], biological [166], technological [191], sociological [28] and cosmological networks [142] as well as power grids [170] and various types of random networks [21].
Of course, network theory is not the only approach to analyze complex systems, therefore one of the general outlook-problems is to draw connections between the network theory and other fields [92, 135, 140], in order to bridge theoretical and applied results of the network techniques. Particularly, the explanation of such connections between the methodology of the flow-networks [254] and the Lagrangian flow-networks [229] could give new insights to the analysis of the flows systems.
In addition, the results for the HOpS model, presented in Chapter IV, could be used to study self-organization in natural complex systems on different scales, from neural networks to transport networks [71, 73, 229, 271].

There are still many related open questions to ask and to answer, which were not included here. For many of such questions I am grateful to a network of collaborators, my teachers and friends, with whom I was lucky enough to work with, and without whom I would be just a lonely node.

List of publications

This dissertation is partly based on the following publications. The identifiers, *e.g.*, P1 or C1, given below are cited in the text to highlight passages that are connected to one or more of these papers.

Papers

P1 H. Bunina L. Tupikina "Authomorphisms of the semigroup of nonnegative invertible matrices of order 2 over rings", Journal of Mathematical Sciences (New York), 183:3, 305â313, (2012)

P2 L. Tupikina, K. Rehfeld, N. Molkenthin, V. Stolbova, N. Marwan, and J. Kurths "Characterizing the evolution of climate networks", in Nonlin. Processes Geophys., 21, 705-711 (2014)

P3 N. Molkenthin, K. Rehfeld V. Stolbova L. Tupikina and J. Kurths "On the influence of spatial sampling on climate networks", Nonlin. Processes Geophys., 21, 651-657 (2014)

P4 J.F. Donges, J. Heitzig, B. Beronov, M. Wiedermann, J. Runge, Q.-Y. Feng, L. Tupikina, V. Stolbova, R.V. Donner, N. Marwan, H.A. Dijkstra, and J. Kurths "Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package", Chaos 25, 113101 (2015)

P5 L.Tupikina, N. Molkenthin, C.Lopez, E.Hernandez-Garcia, N.Marwan and J.Kurths "Correlation networks from flows. The case of forced and time-dependent advection-diffusion dynamics" PLoS ONE 11(4): e0153703 (2016)

P6 H.Kutza, N. Molkenthin, L. Tupikina, N.Marwan, J.Donges, U.Feudel, J. Kurths, R.Donner "A geometric perspective on spatially embedded networks. Quantification of edge anisotropy and application to flow networks", subm. Chaos, arxiv.org/abs/1604.03100

List of frequently used abbreviations

EOF - empirical orthogonal functions
ENSO - El Nino Southern Oscillation
ER - Erdos-Renyi graph
CC - common component
CCEF - common component evolution function
SDS - sequential dynamical system
FCN - functional correlation networks
ADE - Advection-Diffusion equation
CFL-conditions - Courant-Friedrich-Lewy conditions
DN model - Dynamical Network model
HOpS model - Heterogeneous Opinion-Status model

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 27. Mai 2016 Liubov Tupikina